Circular motion

There are many examples of circular motion. We are most interested in those cases where the speed of the object stays constant as it moves in a circular path.

The direction of the acceleration \( \mathbf{a} = \frac{\mathbf{v}}{\mathbf{v}} \) points toward the center of the circle and is called the "centripetal" acceleration. The magnitude of this acceleration is \( a = \frac{v^2}{r} \).

Since Newton's 2nd law tells us that \( F = ma \), then there must be a real force which points in the centripetal (inward) direction that is responsible for this centripetal acceleration.

In this example, that force is the tension in the string. It has to point inward, toward the center of the circle and not outward because "you can't push with a rope".

Circular motion cannot persist without this inward force. When the string breaks, the whirling can moves off in a straight line, tangent to, not outward from, the center of its circular path.

Newton's Gravity

Newton knew that there must be an attractive force on the moon to keep it in the circular orbit around the earth. This force was a centripetal force in that it must point toward the center of the circular path, i.e., toward the earth.

Newton also knew the radius of the moon's orbit about the earth \( R_{\text{em}} = 3.8 \times 10^8 \text{ m} \) and the period of the moon's orbit \( T_m = 27.3 \text{ days} = 2.36 \times 10^6 \text{ s} \).

Calculate the velocity and acceleration of the moon in its circular orbit. (Hint: In one revolution the moon travels a distance equal to the circumference of its circular orbit.)

Recall the ratio of \( a_m \) to \( 9.8 \text{ m/s}^2 \) is \( \frac{a_m}{9.8} \approx \frac{1}{3600} \).

But of course, the moon is a lot farther away from the center of the earth than we are, i.e.,

\[
\frac{d_{\text{em}}}{d_e} = \frac{R_{\text{em}}}{R_e} = 60
\]

So if the acceleration of the moon is \((1/60)^2\) times the acceleration of an apple here on the surface of the earth, then the gravitational force must decrease like the square of the distance from the center of the earth.
Gravity and Distance: The Inverse Square Law

We can better understand how gravity is diluted with distance by considering how paint from a spray can spreads out with increasing distance from the can.

The thickness of the paint layer depends inversely on the square of the distance from the spray can. This inverse square law holds for gravity and other phenomena like the light from a star or the radiation from a piece of uranium.

Newton's Law of Universal Gravitation

Putting all of Newton's ideas together, we can write down a mathematical expression for the Universal force of gravity,

\[ F = \frac{GMm}{d^2} \]

where the constant \( G = 6.7 \times 10^{-11} \text{ N m}^2/\text{kg}^2 \) and \( d \) is the center-to-center distance between the objects. Note that this force also obeys Newton's IIIrd law.

\( g \) decreases with elevation

The gravitational pull (\( F_g \)) here on the surface of the Earth gets smaller as we move away from the surface. Thus the girl's weight (but not her mass) decreases as she climbs up the ladder. If \( h \) is the elevation above the surface of the Earth, then the distance from the center of the Earth is \( d = h + R_e \).

Note that the gravitational force on an object of mass \( m \) at an altitude \( h \) above the Earth's surface is then:

\[ F_g = \frac{GMm}{(h + R_e)^2} = \frac{GMm}{(h + R_e)^2} = mg \]

where \( g = \frac{GM}{d^2} = \frac{GM}{(h + R_e)^2} \)

Calculate the gravitational acceleration of a satellite having an altitude \( h = 6400 \text{ km} = R_e \) above the Earth's surface.

Two spacecraft in outer space attract each other with a force of 400 N. What would be the attractive force be if they were twice as far apart?

Show that the (centripetal) acceleration of the moon due to the gravitational pull of the Earth is independent of the mass of the moon.

But what about \( F = mg \)?

\[ F_g = \frac{GMm}{d^2} = m \left( \frac{GM}{R_e^2} \right) \text{ (here on the Earth's surface)} \]

\[ = m \left( \frac{6.7 \times 10^{-11} \text{ N m}^2/\text{kg}^2 \times 5.98 \times 10^{24} \text{ kg}}{(6.37 \times 10^6 \text{ m})^2} \right) \]

\[ = m(9.8 \text{ N/kg}) = m \left( 9.8 \text{ m/s}^2 \right) = mg_e \]

\( g \) at other locations in the universe

\[ g_e = \left( \frac{GM_e}{R_e^2} \right) \text{ (here on the Earth's surface)} \]

\[ g_m = \left( \frac{GM_m}{R_m^2} \right) \text{ (on the surface of the moon)} = \frac{g_e}{6} \]

If an astronaut in full gear has a weight of 1200 N on the Earth, how much will the astronaut weigh on the moon?

Jupiter is a very large planet with a mass 318 times that of the Earth and a radius that is 11.2 times as large. What is the ratio of \( g \) on the surface of Jupiter to \( g \) here on the surface of the Earth?

The Gravitational Field

We can look at gravity in a different way than just a force acting over a distance. We could instead consider the moon as interacting with the gravitational field of the Earth. Any massive body is the source of a gravitational field. We will use a similar idea for the electric and magnetic fields.
Consider the motion of a satellite with a mass $m$ in a circular orbit of radius $R$ about an object of mass $M$. The period of the orbit is $T$. By substituting Newton's law of gravity

$$F_g = \frac{GmM}{R^2}$$

into Newton's second law of motion, $F = ma$, and then worrying about centripetal acceleration and orbital velocity, one can show that the period and radius of the orbit are related by

$$T^2 = \frac{4\pi^2 R^3}{GM}$$

The higher a satellite's orbit, the longer the orbit takes. The moon needs a full 27.3 days to circle the Earth at a distance of $R = 60R_e$. The International Space Station orbiting at $R = 1.06R_e$ (400 km altitude) takes only 93 min to circle the globe.

Can you have a satellite whose orbital period is exactly $T = 1$ day?