Continuous Time

Consider a continuous time function *x(t)* defined over an interval $0 \le t \le T$. We apply the method of OLS to find the finite sum of the form

$$x_K(t) = a_0 + \sum_{k=1}^{K}\left(a_k \cos k\frac{2\pi}{T}t + b_k \sin k\frac{2\pi}{T}t\right)$$

that best approximates *x(t)*. This sum has a constant term a_0 and sines and cosines at radian frequencies which are integer multiples (harmonics) of $\omega_0 = 2\pi/T$. This ω_0 is the frequency of a sine wave that has exactly one cycle in the interval between 0 and *T*.

We wish to minimize the square error

$$SE_K = \int_0^T \left[x(t) - x_k(t)\right]^2 dt\,.$$

We optimize in the usual way, substituting the first equation into the second and setting derivatives equal to zero.

Consider the following a_3 equation.

$$\frac{\partial SE_K}{\partial a_3} = -2\left[\int_0^T x(t)\cos 3\omega_0 t dt - \hat{a}_0\int_0^T \cos 3\omega_0 t dt - \sum_1^K \hat{a}_k \int_0^T \cos 3\omega_0 t \cos k\omega_0 t dt\right]$$
$$-2\left[\sum_1^K \hat{a}_0 \int_0^T \cos 3\omega_0 t \cos k\omega_0 t dt + \sum_1^K \hat{b}_0 \int_0^T \cos 3\omega_0 t \cos k\omega_0 t dt\right] = 0$$

Conveniently, most of the integrals vanish. For example, when k and m are different integers,

$$\int_0^T \cos k\omega_0 t \cos m\omega_0 t dt = 0$$

because the integrand is a product of two orthogonal functions. When all the algebra is finished, the minimizing coefficients are shown to be:

$$\hat{a}_0 = \frac{1}{T}\int_0^T x(t)dt, \quad \hat{a}_k = \frac{2}{T}\int_0^T x(t)\cos k\omega_0 dt, \text{ and } \hat{b}_k = \frac{2}{T}\int_0^T x(t)\sin k\omega_0 t dt$$

$k = 1, 2, \ldots$

An important property of approximation by sums of harmonic sinusoids is that the SE_K decreases as K increases. In fact,

$$\lim_{K \to \infty} SE_K = 0.$$

Normally we combine the sine and cosine terms into a single sinusoid through the identity:

$$c_i \cos(i\omega_0 t + \varphi_i) = a_i \cos i\omega_0 t + b_i \sin i\omega_0 t$$

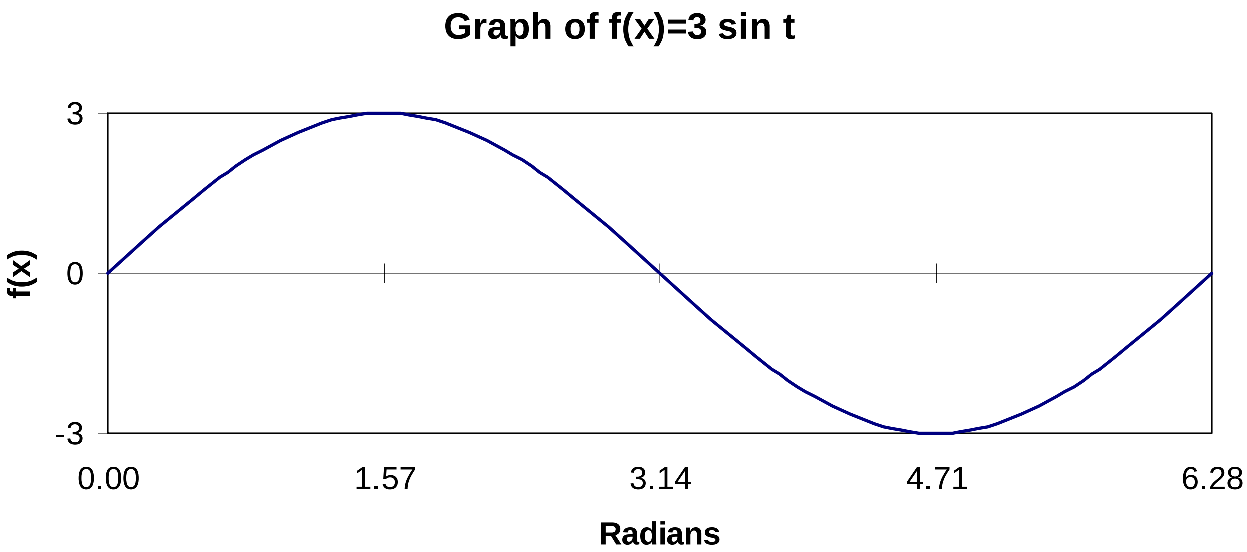
$$\text{with } c_i = \sqrt{a_i^2 + b_i^2} \text{ and } \varphi_i = -\tan^{-1}\left(\frac{b_i}{a_i}\right).$$

The c_i are referred to as the amplitudes, and when the graph of these against k is referred to as the *Amplitude Spectrum.* The φ_i are called phase angles, and the graph of these against k is called the *Phase Spectrum*. Together they form a kind of “catalog” of the sinusoids in the Fourier Series of a given function.

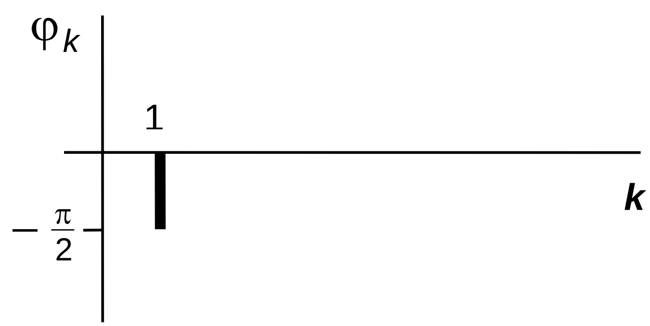
This allows the Fourier Series to be written in the notation

$$x(t) = a_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_0 t + \varphi_i).$$

Consider the following example:



Its spectra look like this:



Complex Fourier Series

$$x(t) = \sum_{-\infty}^{\infty} z_k e^{ik\omega_0 t}$$

in which the complex coefficients z_k are given by

$$\begin{aligned} z_k &= \frac{1}{T}\int_0^T x(t)e^{-i\omega_0 kt}dt \\ &= |z_k|e^{i\angle z_k} \end{aligned}$$

The equivalence with the previous form is given by

$$\begin{aligned} a_0 &= z_0 \\ c_k &= 2|z_k|,\ k = 0,1,-1,2,-2,\ldots \\ \varphi_k &= \angle z_k \\ z_k &= z_{-k}^{*} \end{aligned}$$

This makes $|z_k|$ the amplitude and $\angle z_k = \varphi_k$ the phase.

Discrete Time

The discrete time Fourier Transform pair is:

$$X(\omega) = \sum_{-\infty}^{\infty} x_n e^{-i\omega n}$$

$$x_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{i\omega n} d\omega, \quad n = \ldots, -1, 0, 1, 2, \ldots$$

It can be shown that

$$|X(\omega)| = |X(-\omega)| \quad and$$

$$\angle X(\omega) = -\angle X(-\omega).$$