Traditional Smoothing Techniques

Simple Moving Average:

$$\hat{x}_{t+1} = \sum_{i=1}^{n} \frac{x_{t-i+1}}{n} \quad or \quad \hat{x}_t = \sum_{i=1}^{n} \frac{x_{t-i+1}}{n}$$

Centered Moving Average, assume *n* is odd:

$$\hat{x}_t = \frac{\sum_{i=1}^{(n-1)/2} x_{t-i} + x_t + \sum_{i=1}^{(n-1)/2} x_{t+i}}{n}$$

Weighted Moving Average:

$$\hat{x}_t = \frac{\sum_{i=1}^{n} W_i x_{t-i}}{\sum_{i=1}^{n} W_i}$$

(or, of course, you could set up the W_i so that they simply add to one.)

Note—Linear Moving Averages (MAs of MAs): Consider a system of weights for a 7-point weighted moving average {1,1,1,1,1,1,1}. Another 4-point moving average with weights {1,1,1,1}. Then the 7×4 moving average would have weights {1,2,3,4,4,4,4,3,2,1} and is essentially the convolution of the two sets of weights.

(Single) Exponential Smoothing:

$$\hat{x}_{t+1} = \alpha x_t + (1-\alpha)\hat{x}_t$$

or

$$\hat{x}_t = \alpha x_{t-1} + (1-\alpha)(x_t - x_{t-1})$$

Adaptive Response Rate Single Exponential Smoothing (ARRSES). The advantage here is that α is dynamic.:

$$\hat{x}_{t+1} = \alpha_t x_t + (1+\alpha_t)\hat{x}_t$$

where:

$$e_t = x_t - \hat{x}_t$$

$$E_t = \beta e_t + (1-\beta)E_{t-1}, \text{ smoothed error}$$

$$M_t = \beta|e_t| + (1-\beta)M_{t-1}, \text{ abs.val. of smoothed error}$$

$$\alpha_{t+1} = \left|\frac{E_t}{M_t}\right|$$

$$\beta = 0.2, \text{ a choice variable}$$

Chow's Adaptive Control Method:

- **Can be used for nonstationary data.**
- **α_t is adapted by small increments so as to minimize the MSE.**

$$S_t = \alpha_t x_t + (1-\alpha_t)S_{t-1}$$

$$b_t = \alpha_t(S_t - S_{t-1}) + (1+\alpha_t)b_{t-1}$$

$$\text{and } \hat{x}_{t+1} = S_t + \frac{1-\alpha_t}{\alpha_t} b_t$$

Winters' Linear and Seasonal Exponential Smoothing:

$$\hat{x}_{t+m} = (S_t + b_t m) I_{t-L+m}$$

where

$$S_t = \alpha\left(\frac{x_t}{I_{t-L}}\right) + (1-\alpha)(S_{t-1} + b_{t-1})$$

$$b_t = \gamma(S_t - S_{t-1}) + (1-\gamma) b_{t-1}$$

$$I_t = \beta\left(\frac{x_t}{S_t}\right) + (1-\beta) I_{t-L}$$

L: length of the seasonality
I: seasonal adjustment factor
S_t: smoothed value for the series
m: forecast period
β: a weight to suppress "randomness" (often 0.05)
α: exponential factor for smoothing (often 0.2)
γ: parameter (often 0.1)

DECOMPOSITION METHODS

$$x_t = f(I_t, T_t, C_t, E_t)$$

x_t: actual at time t
I_t: seasonal component (or index) at t
T_t: is trend component at t
C_t: cyclical component at t
E_t: error or random component at t

This function f() can be additive or multiplicative, yielding an <u>additive decomposition</u> or a <u>multiplicative decomposition</u>.

Example, Additive Method:

1. **Compute a moving average of length N, where N is the length of the seasonality. This eliminates the seasonality by averaging seasonally high periods with seasonally low periods, and reduces randomness as well.**

2. **Subtract the moving average from the series. The MA is the trend plus cycle. The error is the seasonal component.**

3. **"Isolate" the seasonal component by averaging them for each of the periods making up the complete length of the seasonality.**

4. **Identify the appropriate form of the trend—linear, exponential, S-curve, etc., and calculate its value at each period t.**

5. **Subtract the estimated "trend" from the deseasonalized series to obtain the cyclical factor.**

6. **Subtract the seasonal, trend, and cycle components from the original series to yield the "random component".**

CENSUS II – X-11
Decomposition/Seasonal Adjustment Method

The Census Method I began in 1954, followed by twelve experimental programs, named X-0, X-1, etc., of Method II. This culminated in X-11.

U. S. Department of Commerce, Bureau of the Census. Julius Shiskin (1955), based upon the ratio-to-moving average classical decomposition.

Shiskin, J., A. H. Young, and J. C. Musgrave. "The X-11 variant of the Census method II seasonal adjustment program." Technical Paper 15, Bureau of the Census, U.S. Department of Commerce, 1967.

Shiskin, Julius. "Seasonal Adjustment of Sensitive Indicators," 1978. In A. Zellner, editor, *Seasonal Analysis of Economic Times Series*, pages 97-103. U. S. Department of Commerce, Bureau of the Census.

X-11 was popular because:

- **It was relatively easy to use.**
- **It did not require restating past values when new data was released.**
- **It handled extreme values well.**
- **It used well-known moving averages methods for estimating trend and seasonal components.**
- **The asymmetric moving averages used near the ends of the time series were thought to be "tried and true".**
- **It had a clear-cut way of estimating trading day effects.**

Statistics Canada extended the method as X-11-ARIMA. This method included the full X-11 method, but used ARIMA backcasts and forecasts to provide optimal estimates of data outside the data window to improve estimates at the "ends" of moving averages.

X-11-ARIMA results in seasonal adjustments whose revisions are smaller, on average, when they are recalculated after future data becomes available.

In the additive decomposition case, extension with optimal forecasts and backcasts for the half-length of the symmetric seasonal filter minimizes revisions in the mean square sense.

Bobbitt, L, and M. C. Otto. "Effects of forecasts on the revisions of seasonally adjusted values using the X-11 seasonal adjustment procedure." In *Proceedings of the Business and Economics Statistics Section*, pages 449-453, Alexandria, Virginia, 1990. American Statistical Association.

X-11-ARIMA also added diagnostics for comparing direct and indirect seasonal adjustments of series that are aggregates of multiple component series.

WE START WITH THE ORIGINAL X-11 METHOD:

Step 1. Trading Day Adjustment

1. **Determine the number of "active" days in each month for the years of interest.**
2. **Compute the average number of trading days for each month.**
3. **Divide the number of days in each month by this average to get an adjustment factor.**
4. **Use the adjustment factor to adjust the monthly figures.**
5. **This creates a value called "original data adjusted for trading days".**

Step 2. Preliminary Seasonal Adjustment.

Seasonality Adjustment

1. **Apply a 12-month MA to eliminate seasonality.**
2. **Average the MAs of 2 successive months to form the 7^{th} month value. This addresses the "centering problem".**
3. **Form the ratio of the original series to the MA series.**

Extreme Values

4. **Calculate the 3x3 month moving average (3-month average of a 3-month average).**
    a. **This is roughly equivalent to a 5-month moving average.**
    b. **Strictly speaking, this should result in the loss of 2 months at the beginning and end of the series, but Census "estimates" replacements for these.**
5. **Calculate the standard deviation of the centered ratios from the 3x3 MA.**
    a. **This is used to construct "control limits" to identify extreme values.**
    b. **If the centered MA > 3x3 MA $\pm$ $2s^2$, then replace it with the average of previous and following period.**

Preliminary Seasonal Factor Estimation & Application

6. **Replace the 6 month at the beginning and end of the ratios by the "nearest" values in a neighboring year.**
7. **"Normalize" years so that the ratios in each year add to 12. (Average ratio is 1.)**
8. **Divide the preliminary seasonal factors into the original data to obtain the preliminary adjusted series.**

Step 3. Refine Seasonal Adjustments.

1. **Apply Spencer's 15 month weighted moving average to the seasonally adjusted data. This is a 5x5x4x4 moving average (quadruple MA)**
   a. **Isolates the "trend-cycle" component.**
2. **Divide the original data by the "trend-cycle component**
   a. **Seasonal and random factors remain. These are called the "final seasonal irregular ratios".**
   b. **Normally Spencer's Method would cause the loss of 7 points at the beginning and the end of the series, so Census replaces the lost data points with estimates.**
3. **Replace the extreme values as above.**
4. **Estimate missing values.**
5. **Adjust (normalize) ratios.**
6. **Take 5-year averages of these final seasonal-irregular ratios**
7. **These are the stable factors (seasonal indices).**

Step 4.

1. **Apply a 3x3 moving average (or 5x5 if it still looks "too" random) to the final seasonal irregular ratios.**
2. **Estimate values for the 2 periods at the beginning and end of the series that would be lost.**
3. **Take the last 2 values for each month, and form an "expected value". For example, for 1992,**

$$\hat{x}_{1992} = \left[\left(\hat{x}_{1991} \times 3\right) - \hat{x}_{1990}\right] / 2$$

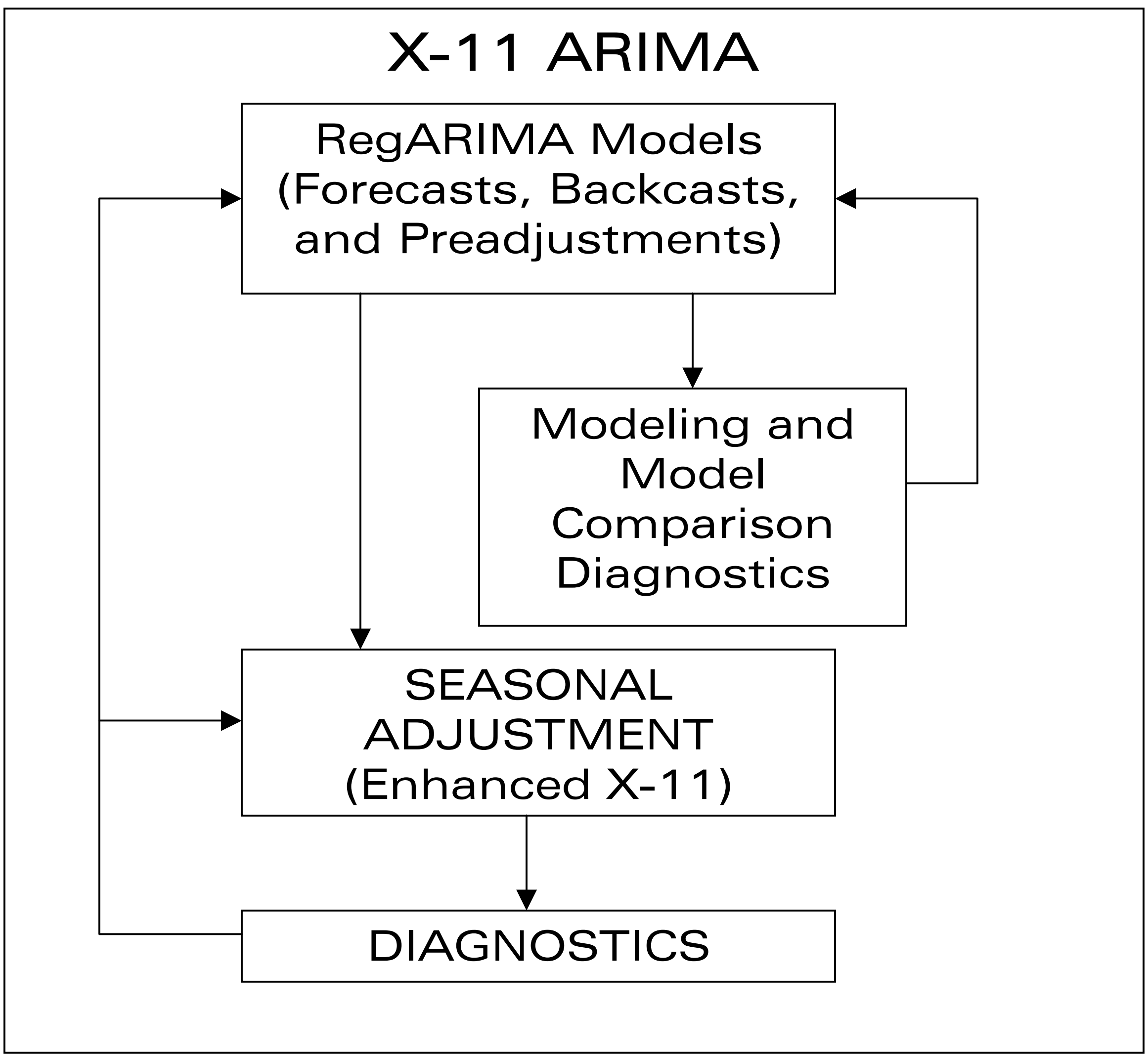
4. **Divide these final seasonal factors into the original data to form the seasonally adjusted series.**

Step 5. Final adjustment.

1. **Calculate a 15 month MA to create the final seasonally adjusted data.**
   a. **This is an estimate of the trend cycle component.**

Step 6. Create a mountain of summary statistics.

The Census X-12-ARIMA includes X-11, but extends the modeling and diagnostic capabilities.



The major methodological improvements of X-12-ARIMA are:

- **New X-11 adjustment options**
- **New diagnostics**
- **New modeling capabilities emphasizing regARIMA modeling . (RegARIMA is a linear regression model with ARIMA time series errors.)**

NEW X-11 ADJUSTMENT OPTIONS

- **New filter options, including:**
  - **longer seasonal moving average,**
  - **user specification of Henderson filters**
  - **modifications to asymmetric moving averages**
- **Option for pseudo-additive decomposition, sometimes useful for series with periodically small or zero values.**
- **Improvements in trading day adjustments and options for user-defined effects based upon preliminary estimates of the irregular component.**

NEW DIAGNOSTIC CAPABILITIES

- **Spectral estimates for detection of seasonal and trading day effects**
- **Revisions history diagnostics for assessing the stability of seasonal adjustments.**
- **Better diagnostics for deciding whether to use direct or indirect adjustments for aggregate series.**

New RegARIMA CAPABILITIES

- **Capability to add regression effects to the models for forecast extension.**
- **Use of RegARIMA models can potentially improve forecasts and backcasts, and provide earlier outlier detection capabilities.**

TYPES OF DECOMPOSITIONS THAT MAY BE SELECTED WHEN USING X-11

- **Multiplicative Decomposition**
  - **Usually appropriate for series of positive values in which the size of the seasonal oscillations increases with the level of the series.**
  - **The seasonally adjusted series is obtained by dividing the original series by the estimated seasonal component.**
- **Additive Decomposition**
  - **More appropriate to stationary series.**
  - **The seasonally adjusted series is obtained by subtracting the estimated seasonal component.**
- **Log-additive Decomposition**
  - **The additive decomposition of the logarithms of the series being adjusted is exponentiated.**
  - **Mainly used for research purposes. Requires a bias correction.**
- **Pseudo-additive decomposition**

In the updated X-11 (in X-12), the Spencer MA is replaced by the Henderson filter. This is either 9, 13, or 23 points and is symmetric. It is designed to approximate a cubic fit to stationary data.

Spectral analysis of the Henderson filter reveals that it has substantial power after the first seasonal frequency (leakage?). As a result, Schips and Stier (1995) argue that the Henderson filter exaggerates short-term cyclical behavior. The 17-term Henderson filter is the shortest

one that does not result in a significant peak beyond the first seasonal frequency.