RECALL THE DESCRIPTION OF $y(t)$ AS THE RESPONSE TO $x(t)$ ACCORDING TO SOME BEHAVIORAL PROCESS $T[\cdot]$:

$$y(t) = T[x(t)]$$

$T[x(t)]$ CAN BE REDUCED TO $g(t)$ SO THAT $y = g(t)$.

SUPPOSE $x(t)$ IS REPEATED (OR A "NEAR COPY" OCCURS) τ PERIODS AFTER THE ONSET OF $x(t)$. THE RESPONSE $y(t)$ COULD BE DESCRIBED AS

$$f_1(t) + a f_1(t-\tau) = f(t)$$

USING GENERALIZED FUNCTIONS, $f(t)$ CAN BE WRITTEN AS

$$f(t) = f_1(t) * [\delta(t) + a\delta(t-\tau)]$$

ACCORDINGLY, IT CAN BE DEMONSTRATED THAT THE POWER SPECTRUM OF THE SIGNAL IS:

$$|F(\omega)|^2 = |F_1(\omega)|^2 [1 + a^2 + 2a\cos(\omega\tau)].$$

THE "ECHO" MANIFESTS AS AN INCREASE IN ENERGY ACCOMPANIED BY A COSINE FUNCTION, THE PERIOD OF WHICH IS THE RECIPROCAL OF THE DELAY TIME τ, RIDING ON THE ENVELOPE OF THE POWER SPECTRUM.

IF THE "FREQUENCY CONTENT" OF THE "PRIMARY" AND THE "ECHO" ARE DIFFERENT, THEN

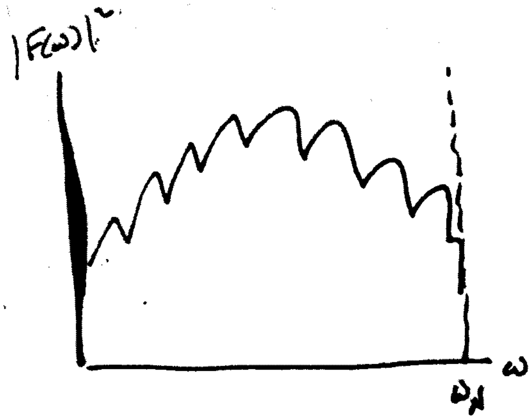
$$f(t) = f_1(t) + f_2(t-\tau)$$

AND

$$|F(\omega)|^2 = |F_1(\omega)|^2 + 2|F_1(\omega)||F_2(\omega)| \cos[\phi_1(\omega) - \phi_2(\omega) - \omega\tau].$$

THE MODULATING COSINE IS PHASE SHIFTED BY AMOUNT EQUAL TO THE DIFFERENCE IN THE PHASES OF THE PRIMARY AND ITS "ECHO" AT EACH FREQUENCY,

$$\phi_1(\omega) - \phi_2(\omega).$$



QUESTION: HOW TO SEPARATE $f_1(t)$ FROM ITS ECHO?

CEPSTRUM: HOMOMORPHIC DECONVOLUTION (de-convolve the wavetrain)

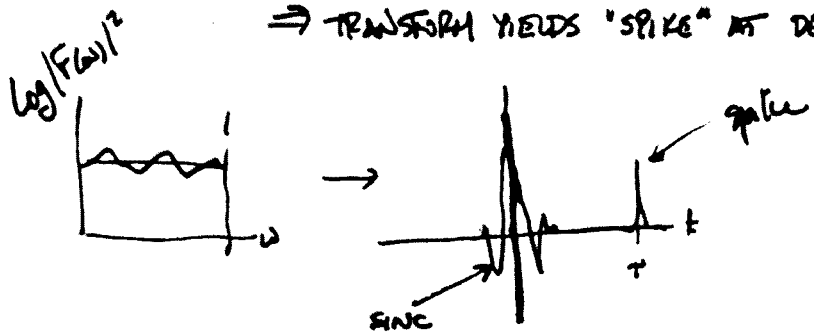
CEPSTRUM (Def.): CLASS OF INTEGRAL TRANSFORMS WHOSE KERNEL IS A FUNCTION OF THE Z-TRANSFORM OF A REAL SEQUENCE.

POISSON (1823), SCHWARZ (1872), SZEGÖ (1915), KOLMOGOROV (1939)
BOGERT (1963) et al.

BOGERT et al.:

IF THE ENVELOPE OF $|F(\omega)|^2$ COULD BE MADE "WHITE ENOUGH", IT WOULD APPROXIMATE A BOXCAR FUNCTION. ITS Z-TRANSFORM WOULD BE A SINC FUNCTION AT THE ORIGIN,

MODULATION PATTERN ON THE POWER SPECTRUM = SIMPLE COSINE
$\Rightarrow$ TRANSFORM YIELDS "SPIKE" AT DELAY τ



$\Longleftrightarrow$ POWER CEPSTRUM

(IGNORES THE PHASE)

- OPPENHEIM et al. (1968), Noll (1964, 1967), Kemerait & Childers (1971, 1972)

INCORPORATE PHASE-SPECTRAL INFORMATION BY REPLACING THE (REAL) LOG OF THE POWER SPECTRUM WITH COMPLEX LOG OF COMPLEX FOURIER SPECTRUM

$\Rightarrow$ COMPLEX CEPSTRUM.

- INVERTIBLE (CONTAINS "COMPLETE" INFORMATION ON $f(t)$)
- COMBINED WITH LINEAR FILTERS, CAN BE USED FOR ECHO EXTRACTION & RECOVERY.

$\Rightarrow$ HOMOMORPHIC DECONVOLUTION

ASSUMPTIONS:

- ECHO IS A "REASONABLY GOOD" COPY OF PRIMARY
- COMPUTATIONAL PROBLEMS CAN BE OVERCOME (?)

POWER CEPSTRUM OF SERIES $x(nT)$

$$x_{pc}(nT) = \left| \frac{1}{2\pi i} \oint_c \log |X(z)| \, z^{n-1} dz \right|^2$$

$$\text{let } x' = \frac{1}{2\pi i} \oint_c \log |X(z)| \, z^{n-1} dz$$

COSINE-SQUARED CEPSTRUM:

$$x_{cs}(nT) = \operatorname{sgn}\{ Re[x'] \} \, x_{pc}(nT)$$

THE ADDITION OF THE SIGNUM FUNCTION ALLOWS THE CEPSTRUM TO DETERMINE NOT ONLY THE ECHO ARRIVAL TIME, BUT ITS POLARITY AS WELL.

☆ BECAUSE $x(nT)$ IN ECONOMICS ARE REAL, FINITE SEQUENCES, THE ANNULUS OF CONVERGENCE OF $X(z)$ ALWAYS INCLUDES THE UNIT CIRCLE. THEREFORE, THE TRANSFORMS MAY BE COMPUTED AS FFTs.

$\left(\log X(\omega) = \log |X(\omega)| + i\,\phi_{cd}(\omega) \right)$

ϕ_{cd} = continuous, detrended

COMPLEX CEPSTRUM

$$\hat{x}(nT) = \frac{1}{2\pi i}\oint_C \log[X(z)]z^{n-1}dz$$

where

$$\hat{x}(0) = \log[x(0)]$$

(1) windowing, clipping, filtering
(2) Compute FFT
(3) Compute amplitude & phase spectra (ampl. spec. optionally smoothed)
(4) take natural log of ampl. spectrum
(5) phase spectrum made continuous
~~(6)~~

to be homomorphic, the cepstrum can only be unique if $\log|X(z)|$ is analytic in an annular region containing contour C; this condition is achieved when the phase is made continuous

(4) The z-transform $X(z)$ may have zeros outside the unit circle, so that $X(z)$ can be represented by

$$X(z) = A z^{-m_o}\prod_{k=1}^{m_i}(1-a_k z^{-1})\prod_{k=1}^{m_o}(1-b_k z)$$

where the a_k's are the m_i zeros inside the unit circle, the b_k's are the m_o zeros outside the unit circle, and

$$|a_k|<1 \text{ and } |b_k|<1$$

z^{-m_o} shift of the input sequence.

Contour C is $z = e^{\delta + i\omega}$

$$\hat{x}(nT) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log[X(e^{\sigma+i\omega})]\, e^{\sigma n T} e^{i\omega n T}\, d\omega$$

Let the contribution of z^{-m_0} to $\hat{x}(nT)$ be $\hat{\phi}(nT)$
contour of int. be the unit circle, then

$$\hat{\phi}(nT) = \frac{1}{2\pi}\int_{-\pi}^{\pi} m_0 \log[e^{i\omega}]\, e^{i\omega n T}\, d\omega$$

which integrates to

$$\hat{\phi}(nT) = -\frac{m_0 \cos \pi n T}{nT}$$

Often $X(z)$ will have many zeros outside the unit circle; thus m_0 is large and the effect of $\hat{\phi}(nT)$ is a severe "ringing" which completely dominates the cepstrum, cloaking all information.

⇒ Remove the linear phase component.

(7) Construct complex function:

$$\cancel{\boxed{\hat{F}(\omega)}} \quad \hat{F}(\omega) = \log A(\omega) + i\,\phi_{cd}(\omega)$$

inverse FFT on $\hat{F}(\omega)$

$$\hat{x}_I = 0\ , \quad \hat{x}_R(k) = \hat{x}_R(k) = \text{complex cepstrum}$$

(8) Comb Filter

(9) Reverse process