VAR'S – THEORY

CONSIDER BIVARIATE GRANGER CAUSALITY TESTS:

IN A BIVARIATE VAR DESCRIBING x and y, y DOES NOT GRANGER-CAUSE x IF THE COEFFICIENT MATRICES Φ_j ARE ALL LOWER TRIANGULAR FOR ALL j:

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \phi_{11}^{(1)} & 0 \\ \phi_{21}^{(1)} & \phi_{22}^{(1)} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} \phi_{11}^{(2)} & 0 \\ \phi_{21}^{(2)} & \phi_{22}^{(2)} \end{bmatrix} \begin{bmatrix} x_{t-2} \\ y_{t-2} \end{bmatrix} + \cdots + \begin{bmatrix} \phi_{11}^{(p)} & 0 \\ \phi_{21}^{(p)} & \phi_{22}^{(p)} \end{bmatrix} \begin{bmatrix} x_{t-p} \\ y_{t-p} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}$$

CLEARLY THE ZERO IN THE UPPER RIGHT CORNER IMPLIES THAT OPTIMAL ONE-PERIOD AHEAD FORECASTS OF x DO NOT DEPEND ON LAGGED y.

NOTE: THE ZERO (OR A ZERO RESTRICTION) IMPLIES CONFIDENCE IN A CASUAL RELATIONSHIP.

NEXT: ASSUME x_t IS COVARIANCE STATIONARY OF LENGTH N. BY THE WOLD DECOMPOSITION THEOREM (1938), x_t CAN BE REPRESENTED BY

$$x_t = \mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$$

WHERE ε_t IS A WHITE NOISE ERROR PROCESS, $\psi_0 = 1$, AND $\sum_{j=0}^{\infty} \psi_j^2 < \infty$.

THIS CAN BE GENERALIZED IF x_t IS A COVARIANCE STATIONARY VECTOR OF LENGTH N, THEN x_t HAS A MOVING AVERAGE REPRESENTATION BY THE WOLD DECOMPOSITION THEOREM.

IF THIS IS TRUE, THEN CERTAINLY x_t CAN BE WRITTEN AS:

(1) $$x_t = A(L)x_t + B(L)\varepsilon_t$$

WHERE $A(L)$ AND $B(L)$ ARE MATRICES OF POLYNOMIALS IN THE LAG OPERATOR, AND THE COEFFICIENT ON L^0 IS ZERO FOR ALL ELEMENTS OF $A(L)$.

IF $B(L)$ IS INVERTIBLE, THEN (1) CAN BE WRITTEN AS A VAR:

(2) $$x_t = C^*(L)x_t + \varepsilon_t .$$

NOTE: THERE ARE ONLY LAGGED VALUES OF x ON THE RHS; THE COEFFICIENT ON L^0 IS ZERO FOR ALL ELEMENTS OF $C^*(L)$.

IN THEORY THE ELEMENTS OF C^* ARE POLYNOMIALS THAT MAY BE OF INFINITE DEGREE, BUT IN PRACTICE THE LAGGED IS TRUNCATED TO SOME FINITE LENGTH M.

HENCE:

(3) $$x_t = C(L)x_t + \varepsilon_t = \sum_{s=1}^{M} C_s x_{t-s} + \varepsilon_t .$$

C_s IS A $N \times N$ COEFFICIENT MATRIX ASSOCIATED WITH THE sTH LAG.

(THE VAR HAS N EQUATIONS IN WHICH EACH ELEMENT OF x_t IS A LINEAR FUNCTION OF ITS OWN LAGGED VALUES AND THE LAGGED VALUES OF ALL THE OTHER VARIABLES.)

THE MODEL (3) CAN BE CONSISTENTLY AND EFFICIENTLY ESTIMATED BY APPLYING OLS TO EACH EQUATION.

IT IS THE EQUIVALENT MOVING AVERAGE REPRESENTATION THAT IS USED TO EXAMINE THE DYNAMIC RESPONSE TO SHOCKS

IF $[I - C(L)]$ HAS AN INVERSE, THEN (3) CAN BE WRITTEN AS

$$(4) \quad x_t = [I - C(L)]^{-1} \varepsilon_t = D(L)\varepsilon_t = \sum_{s=0}^{\infty} D_s \varepsilon_{t-s}$$

WHERE D_s IS $N \times N$.

FROM (4) IF $\exists$ NO CONTEMPORANEOUS CORRELATION AMONG THE ELEMENTS OF ε_t, d_s^{ij}, the i,j th ELEMENT OF D_s TRACES OUT THE RESPONSE OF THE ith ELEMENT OF x_t OVER TIME(s) TO AN INNOVATION IN THE jth ELEMENT OF x_t.

☆ NORMALLY THESE "INNOVATIONS" WILL BE CONT. CORRELATED, THEREFORE IT IS IMPOSSIBLE TO UNIQUELY DECOMPOSE THE VARIANCE OF EACH VARIABLE INTO COMPONENTS ACCOUNTED FOR BY EACH INNOVATION.

SOLUTION REQUIRES A TRIANGULAR ORTHOGONALIZATION TRANSFORMATION TO THE VECTOR OF INNOVATIONS, ε_t, TO PRODUCE A NEW VECTOR OF INNOVATIONS HAVING THE IDENTITY MATRIX AS ITS COVARIANCE MATRIX.

WE WANT A LOWER TRIANGULAR MATRIX G SO THAT

$$V_t = G\varepsilon_t \text{ AND THE}$$

COVARIANCE MATRIX OF V_t IS THE IDENTITY MATRIX.

ONCE WE FIND G,

$$(5) \quad x_t = D(L)G^{-1}G\varepsilon_t = F(L)v_t = \sum_{s=0}^{\infty} F_s v_{t-s}$$

THE TRANSFORMATION G IS NOT UNIQUE AND ESSENTIALLY IMPOSES A PARTICULAR CAUSAL ORDERING ON THE VARIABLES

G IS LOWER TRIANGULAR, SO G^{-1} IS UPPER TRIANGULAR. $\therefore$ G^{-1} CREATES THE ZERO RESTRICTIONS THAT IMPLY CAUSALITY.

TRIANGULAR DECOMPOSITION

WHEN WE PERFORM GAUSSIAN ELIMINATION ON A MATRIX "A", IN THE "k-th" STEP, WE FORM A^{k+1} FROM A^k BY SUBTRACTING MULTIPLES OF THE "k-th" ROW FROM ROWS $k+1$ TO N.

THIS CAN BE WRITTEN $A^{(k+1)} = M_k A^{(k)}$.

M_k IS NONSINGULAR, SO IT FOLLOWS THAT

$$A^{(k)} = M_k^{-1} A^{(k+1)}.$$

BY INDUCTION

$$A^{(1)} = M_1^{-1} M_2^{-1} \cdots M_{N-1}^{-1} A^{(N)}$$

IT CAN BE SHOWN THAT

$$M_1^{-1} M_2^{-1} \cdots M_{N-1}^{-1} = L$$ IS LOWER TRIANGULAR WITH UNITARY DIAGONAL ELEMENTS

BUT $A^{(1)} = A$, $A^{(N)}$ IS UPPER TRIANGULAR, SO

$$A = LU \quad , \quad U = A^{(N)}$$

SO GAUSSIAN ELIMINATION IS EQUIVALENT TO A TRIANGULAR DECOMPOSITION.

IF A IS SYMMETRIC, THEN THIS DECOMPOSITION DESTROYS THE SYMMETRY. FOR EXAMPLE,

$$\begin{bmatrix} 16 & 4 & 8 \\ 4 & 5 & -4 \\ 8 & -4 & 22 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 0.25 & 1 & \\ 0.5 & -1.5 & 1 \end{bmatrix} \begin{bmatrix} 16 & 4 & 8 \\ & 4 & -6 \\ & & 9 \end{bmatrix}$$

A SLIGHTLY DIFFERENT FORM OF DECOMPOSITION IS

$$A = L D L^T$$

USING THE SAME EXAMPLE,

$$\begin{bmatrix} 16 & 4 & 8 \\ 4 & 5 & -4 \\ 8 & -4 & 22 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 0.25 & 1 & \\ 0.5 & -1.5 & 1 \end{bmatrix} \begin{bmatrix} 16 & & \\ & 4 & \\ & & 9 \end{bmatrix} \begin{bmatrix} 1 & 0.25 & 0.5 \\ & 1 & -1.5 \\ & & 1 \end{bmatrix}$$

WE COULD CONSTRUCT A MATRIX $D^{1/2} = [\sqrt{d_i}]$

SO THAT $D = D^{1/2} D^{1/2}$

THIS ALLOWS

$$A = L D L^T = L D^{1/2} D^{1/2} L^T = \tilde{L} \tilde{L}^T$$

where $\tilde{L} = L D^{1/2}$

THE DECOMPOSITION $A = \tilde{L}\tilde{L}^T$ IS THE <u>CHOLESKI DECOMPOSITION</u>.
(IF A IS NOT P.D., $\tilde{L}$ CONTAINS COMPLEX ELEMENTS)

$$\begin{bmatrix} 16 & 4 & 8 \\ 4 & 5 & -4 \\ 8 & -4 & 22 \end{bmatrix} = \begin{bmatrix} 4 & & \\ 1 & 2 & \\ 2 & -3 & 3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 2 \\ & 2 & -3 \\ & & 3 \end{bmatrix}$$

SIDE NOTE: THE CHOLESKI DECOMPOSITION TAKES ABOUT 1/2 THE COMPUTATIONS THAT THE TRIANGULAR (GAUSSIAN) DECOMPOSITION.

THE FORMULAE FOR THE CHOLESKI DECOMPOSITION ARE:

$$l_{ii} = \left(a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2 \right)^{1/2}$$

$$l_{ij} = \frac{a_{ij} - \sum_{k=1}^{j-1} l_{ik} l_{jk}}{l_{jj}} \quad , j < i$$

☆ NOTICE: REORDERING THE MATRIX ROWS OR COLUMNS RESULTS IN A DIFFERENT DECOMPOSITION.

WHEN WE TRACE OUT THE RESPONSE OF AN ELEMENT OF X_t TO AN ORTHOGONALIZED INNOVATION, THIS IS THE <u>IMPULSE</u> <u>RESPONSE</u> AND REMAINS UNIQUE.

SINCE WE HAVE ORTHOGONALIZED THE INNOVATIONS, THE VARIANCE OF THE FORECAST ERROR FOR ANY VARIABLE IS DECOMPOSED INTO THE SUM OF THE CONTRIBUTIONS FROM EACH OF THE INNOVATIONS IN THE SYSTEM <u>ACCORDING TO THE ORTHOGONALIZATION</u>.

THIS IS THE <u>VARIANCE</u> <u>DECOMPOSITION</u>.

☆ MOST RESEARCHERS ARGUE THAT ONLY <u>VARIANCE</u> DECOMPOSITION RESULTS THAT ARE ROBUST TO RE-ORDERING CAN BE CONSIDERED SIGNIFICANT.

KANG (1985) HAS ARGUED THAT (VAR) GRANGER TESTS ARE QUITE SENSITIVE TO METHOD OF DETRENDING.

IT APPEARS THAT ORDERING IS LESS IMPORTANT FOR HIGH FREQUENCY DATA, LESS FOR TEMPORALLY AGGREGATED DATA.

LAG LENGTHS MAY ALSO BE IMPORTANT.