

The z-Transform

ECON 397

Macroeconometrics

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z-Transform

- The **z-transform** is the most general concept for the transformation of discrete-time series.
- The **Laplace transform** is the more general concept for the transformation of continuous time processes.
- For example, the Laplace transform allows you to transform a differential equation, and its corresponding initial and boundary value problems, into a space in which the equation can be solved by ordinary algebra.
- The switching of spaces to transform calculus problems into algebraic operations on transforms is called operational calculus. The Laplace and z transforms are the most important methods for this purpose.

The Transforms

The Laplace transform of a function $f(t)$:

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt$$

The one-sided z-transform of a function $x(n)$:

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$$

The two-sided z-transform of a function $x(n)$:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Relationship to Fourier Transform

Note that expressing the complex variable z in polar form reveals the relationship to the Fourier transform:

$$X(re^{i\omega}) = \sum_{n=-\infty}^{\infty} x(n)(re^{i\omega})^{-n}, \text{ or}$$

$$X(re^{i\omega}) = \sum_{n=-\infty}^{\infty} x(n)r^{-n}e^{-i\omega n}, \text{ and if } r = 1,$$

$$X(e^{i\omega}) = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-i\omega n}$$

which is the **Fourier transform** of $x(n)$.

Region of Convergence

The z-transform of $x(n)$ can be viewed as the Fourier transform of $x(n)$ multiplied by an exponential sequence r^n , and the z-transform may converge even when the Fourier transform does not.

By redefining convergence, it is possible that the Fourier transform may converge when the z-transform does not.

For the Fourier transform to converge, the sequence must have finite energy, or:

$$\sum_{n=-\infty}^{\infty} |x(n)r^{-n}| < \infty$$

Convergence, continued

The power series for the z-transform is called a **Laurent series**:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

The Laurent series, and therefore the z-transform, represents an analytic function at every point inside the region of convergence, and therefore the z-transform and all its derivatives must be continuous functions of z inside the region of convergence.

In general, the Laurent series will converge in an annular region of the z-plane.

Some Special Functions

First we introduce the **Dirac delta function** (or unit sample function):

$$\delta(n) = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases} \quad \text{or} \quad \delta(t) = \begin{cases} 0, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

This allows an arbitrary sequence $x(n)$ or continuous-time function $f(t)$ to be expressed as:

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

$$f(t) = \int_{-\infty}^{\infty} f(x)\delta(x-t)dt$$

Convolution, Unit Step

These are referred to as discrete-time or continuous-time **convolution**, and are denoted by:

$$x(n) = x(n) * \delta(n)$$

$$f(t) = f(t) * \delta(t)$$

We also introduce the **unit step function**:

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \quad \text{or} \quad u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Note also:

$$u(n) = \sum_{k=-\infty}^{\infty} \delta(k)$$

Poles and Zeros

When $X(z)$ is a rational function, i.e., a ratio of polynomials in z , then:

1. The roots of the numerator polynomial are referred to as **the zeros of $X(z)$** , and
2. The roots of the denominator polynomial are referred to as **the poles of $X(z)$** .

Note that no poles of $X(z)$ can occur within the region of convergence since the z -transform does not converge at a pole.

Furthermore, the region of convergence is bounded by poles.

Example

$$x(n) = a^n u(n)$$

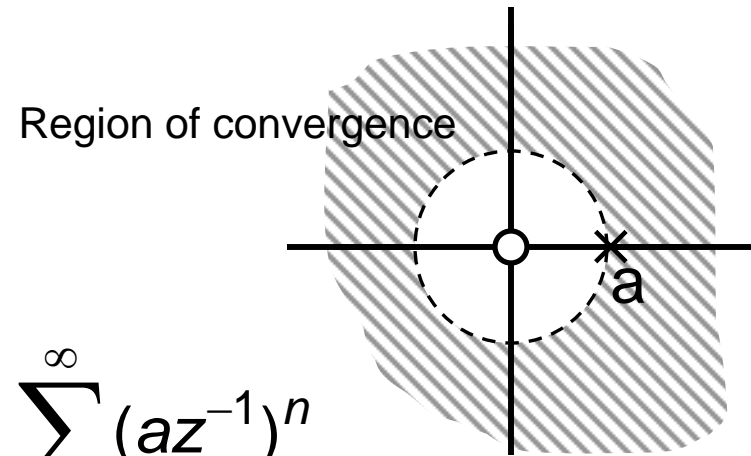
The z-transform is given by:

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

Which converges to:

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a} \quad \text{for } |z| > |a|$$

Clearly, $X(z)$ has a zero at $z = 0$ and a pole at $z = a$.



Convergence of Finite Sequences

Suppose that only a finite number of sequence values are nonzero, so that:

$$X(z) = \sum_{n=n_1}^{n_2} x(n)z^{-n}$$

Where n_1 and n_2 are finite integers. Convergence requires

$$|x(n)| < \infty \text{ for } n_1 \leq n \leq n_2.$$

So that finite-length sequences have a region of convergence that is at least $0 < |z| < \infty$, and may include either $z = 0$ or $z = \infty$.

Inverse z-Transform

The inverse z-transform can be derived by using Cauchy's integral theorem. Start with the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Multiply both sides by z^{k-1} and integrate with a contour integral for which the contour of integration encloses the origin and lies entirely within the region of convergence of $X(z)$:

$$\begin{aligned} \frac{1}{2\pi i} \oint_C X(z)z^{k-1} dz &= \frac{1}{2\pi i} \oint_C \sum_{n=-\infty}^{\infty} x(n)z^{-n+k-1} dz \\ &= \sum_{n=-\infty}^{\infty} x(n) \frac{1}{2\pi i} \oint_C z^{-n+k-1} dz \end{aligned}$$

$$\frac{1}{2\pi i} \oint_C X(z)z^{k-1} dz = x(n) \text{ is the inverse z - transform.}$$

Properties

- z-transforms are linear:

$$\mathcal{Z}[ax(n) + by(n)] = aX(z) + bY(z)$$

- The transform of a shifted sequence:

$$\mathcal{Z}[x(n + n_0)] = z^{n_0} X(z)$$

- Multiplication:

$$\mathcal{Z}[a^n x(n)] = Z(a^{-1}z)$$

But multiplication will affect the region of convergence and all the pole-zero locations will be scaled by a factor of a .

Convolution of Sequences

$$w(n) = \sum_{k=-\infty}^{\infty} x(k)y(n-k)$$

Then

$$\begin{aligned} W(z) &= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k)y(n-k) \right] z^{-n} \\ &= \sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} y(n-k) z^{-n} \end{aligned}$$

let $m = n - k$

$$W(z) = \sum_{k=-\infty}^{\infty} x(k) \left[\sum_{m=-\infty}^{\infty} y(m)z^{-m} \right] z^{-k}$$

$W(z) = X(z)Y(z)$ for values of z inside the regions of convergence of both.

More Definitions

Definition. **Periodic**. A sequence $x(n)$ is **periodic with period λ** if and only if $x(n) = x(n + \lambda)$ for all n .

Definition. **Shift invariant** or **time-invariant**. Consider a sequence $y(n)$ as the result of a transformation T of $x(n)$. Another interpretation is that T is a **system** that responds to an **input** or **stimulus** $x(n)$:

$$y(n) = T[x(n)].$$

The transformation T is said to be **shift-invariant** or **time-invariant** if:

$$y(n) = T[x(n)] \text{ implies that } y(n - k) = T[x(n - k)]$$

For all k . “Shift invariant” is the same thing as “time invariant” when n is time (t).

Let $h_k(n)$ be the response of the system to $\delta(n - k)$, a "spike" or shock occurring at $n = k$. Then :

$$y(n) = T \left[\sum_{k=-\infty}^{\infty} x(k) \delta(n - k) \right] = \sum_{k=-\infty}^{\infty} x(k) T[\delta(n - k)]$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h_k(n).$$

If we have time invariance of the transform T , then

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n - k) = x(n) * h(n).$$

This implies that the system can be completely characterized by its impulse response $h(n)$. This obviously hinges on the stationarity of the series.

Definition. Stable System. A system is stable if

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

Which means that a bounded input **will not** yield an unbounded output.

Definition. Causal System. A **causal system** is one in which changes in output do not precede changes in input. In other words,

$$\begin{aligned} &\text{If } x_1(n) = x_2(n) \text{ for } n \leq n_0 \\ &\text{then } T[x_1(n)] = T[x_2(n)] \text{ for } n < n_0. \end{aligned}$$

Linear, shift-invariant systems are causal iff $h(n) = 0$ for $n < 0$.

Given $y(n) = \sum_{k=-\infty}^{\infty} x(k)h_k(n)$ let $x(n)$ be sinusoidal. That is,

let $x(n) = e^{i\omega n}$ for $-\infty < n < \infty$. Then

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)e^{-i\omega(n-k)} = e^{i\omega n} \sum_{k=-\infty}^{\infty} h(k)e^{-i\omega k}$$

Let $H(e^{i\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-i\omega k}$ so that

$$y(n) = H(e^{i\omega})e^{i\omega n}.$$

Here $H(e^{i\omega})$ is called the **frequency response** of the system whose **impulse response** is $h(n)$. Note that $H(e^{i\omega})$ is the Fourier transform of $h(n)$.

We can generalize this state that:

$$X(e^{i\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-i\omega n}$$

These are the Fourier transform pair.

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{i\omega}) e^{i\omega n} d\omega$$

If $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$, then the transform is absolutely convergent and

converges uniformly to a continuous function of ω .

This implies that the frequency response of a stable system always converges, and the Fourier transform exists.

If $x(n)$ is constructed from some continuous function $x_c(t)$ by sampling at regular periods T (called “**the sampling period**”), then $x(n) = x_c(nT)$ and $1/T$ is called the **sampling frequency** or **sampling rate**.

If ω_0 is the highest radial frequency of sinusoids comprising $x(nT)$, then

$$\omega_0 < \frac{2\pi}{T} \quad \text{or} \quad \frac{1}{T} > \frac{\omega_0}{2\pi}$$

Is the sampling rate required to guarantee that $x_c(nT)$ can be used to fully recover $x_c(t)$, This sampling rate ω_0 is called the **Nyquist rate** (or frequency). Sampling at less than this rate will involve losing information from the time series.

Assume that the sampling rate is at least the Nyquist rate.

$$X(e^{i\omega T}) = \frac{1}{T} X_c(i\omega), \quad -\frac{\pi}{T} \leq \omega \leq \frac{\pi}{T}$$

From the continuous time Fourier transform :

$$x_c(t) = \frac{1}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} X_c(i\omega) e^{i\omega t} d\omega.$$

Combining :

$$x_c(t) = \frac{1}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} TX(e^{i\omega T}) e^{i\omega t} d\omega.$$

Since $X(e^{i\omega T}) = \sum_{k=-\infty}^{\infty} x_c(kT) e^{-i\omega Tk}$, we have

$$x_c(t) = \frac{T}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \left[\sum_{k=-\infty}^{\infty} x_c(kT) e^{-i\omega Tk} \right] e^{i\omega t} d\omega$$

Changing the order of summation and integration,

$$x_c(t) = \sum_{k=-\infty}^{\infty} x_c(kT) \left[\frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} e^{i\omega(t-kT)} d\omega \right]$$

Evaluating the integral :

$$x_c(t) = \sum_{k=-\infty}^{\infty} x_c(kT) \frac{\sin\left(\frac{\pi}{T}(t-kT)\right)}{\left(\frac{\pi}{T}(t-kT)\right)}$$

NOTE: This equation allows for recovering the continuous time series from its samples. This is valid only for bandlimited functions.