CAUCHY-GOURSAT THEOREM.

One of the most important results of complex analysis is the Cauchy-Goursat theorem, alternately called Cauchy's Integral Theorem. The reason for this ambiguity in naming is due to the fact that Cauchy performed the original work, while Edward Goursat removed from the hypotheses one of the most important restrictions to its application. One of the simplest approaches to this theory is via the deformation of simple-closed contours which follows.

Given two line segments of different lengths, we say that one can be continuously deformed into the latter if an only if a function can be found which maps from one to the other that is

- a one-to-one correspondence (bi-unique), and
- bicontinuous.

Observe in the figure below the line segments $\overline{AB}$ and $\overline{CD}$, where we have $AB < CD$. We first draw the auxiliary rays $CA$ and $DB$ until their intersection at point $P$, and note that these rays intuitively denote a one-to-one correspondence between the endpoints of the two segments $A$ and $C$, and $B$ and $D$. Furthermore, it can be seen that given any point on $\overline{AB}$ called $M$, we can find the corresponding point on $\overline{CD}$ by noting the point $N$ at which $PM$ intersects $CD$.

This is a continuous mapping of $\overline{AB}$ onto $\overline{CD}$ because points close together on $\overline{AB}$ are mapped close together on $\overline{CD}$. Conversely, points close together on $\overline{CD}$ are mapped onto points close together on $\overline{AB}$. Therefore, the figure demonstrates a simple example of a bi-unique, bicontinuous correspondence between the two point sets, which is the requirement for calling the two sets (segments), topologically equivalent, homomorphic, or alternately homotopic, which are the proper terms for continuously deformable.
We say a loop $\Gamma_0$ can be continuously deformed into the loop $\Gamma_1$ in the domain $D$ if $\Gamma_0$ (considered as an elastic string with indicated direction) can be continuously moved about the plane (or stretched, or contracted) without leaving $D$ in such a manner that is ultimately coincides with $\Gamma_1$ (in position as well as orientation). (See the figure below).

![Diagram showing loops $\Gamma_0$, $\Gamma_1$, and $\Gamma_2$ in a domain $D$.]

We formalize in the following:

**DEFINITION:** The loop $\Gamma_0$ is said to be *continuously deformable* (topologically equivalent, homeomorphic, or homotopic) to the loop $\Gamma_1$ in the domain $D$ if there exists a function $z(S,t)$ continuous on the unit square $0 \leq S \leq 1$, $0 \leq t \leq 1$, which satisfies the following conditions:

- For each fixed $S$ in $0 \leq S \leq 1$, the function $z(S,t)$ parametrizes a loop lying in $D$,
- The function $z(0,t)$ parametrizes the loop $\Gamma_0$, and
- The function $z(1,t)$ parametrizes the loop $\Gamma_1$.

It turns out that all loops can be continuously deformed to points in a certain kind of domain. It is defined below.

**DEFINITION:** Any domain $D$ possessing the property that every loop in $D$ can be continuously deformed in $D$ to a point is called a *simply connected domain*. Roughly speaking, we say that a simply connected domain has no "holes", for if there were a hole in $D$, then a loop surrounding it could not be shrunk to a point without leaving $D$. 
EXAMPLES:

SIMPLY CONNECTED  SIMPLY CONNECTED  NOT SIMPLY CONNECTED (MULTIPLY CONNECTED)

We must state one final result before stating the theorem which is the main purpose of this section. We do so without proof, since it involves procedures far afield from the basic techniques of complex analysis.

Theorem. Deformations Invariance Theorem

let $f(z)$ be a function analytic in a domain $D$ continuing the loops $\Gamma_0$ and $\Gamma_1$. If these loops can be continuously deformed into one another in $D$, then

$$\int_{\Gamma_0} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz$$

And now, an easy consequence is

THE CAUCHY-GOURSAT THEOREM.

If $f(z)$ is analytic in a simply connected domain $D$ and $\Gamma$ is any loop (closed contour) in $D$, then

$$\int_{\Gamma} f(z)\,dz = 0.$$ 

PROOF: The proof is immediate, in a simply connected domain any loop can be shrunk to a point, the integral of $f(z)$ over a loop consisting of a single point is, of course, zero.

As a concluding summary, we state the following:

Theorem. In a simply connected domain, an analytic function has an antiderivative, its contour integrals are independent of path, and its loop integrals equal zero.
Example:
\[ \oint \frac{e^z}{z^2 - 9} \, dz \]
\[ |z| = 2 \]

The notation signifies that the contour of integration is a loop, and, in particular, is the circle \(|z|=2\) traversed once in the counter clockwise direction.

\[ |z|=2 \]

the integral \(\frac{e^z}{z^2 - 9}\) is analytic everywhere except at \(z=\pm3\),

that is where the denominator equals zero. Thus from the figure we see that the contour can be shrunk to a point in the domain of analyticity, and thus

\[ \oint \frac{e^z}{z^2 - 9} \, dz = 0 \]  \hspace{1cm} \text{(by Cauchy's Theorem).}
CAUCHY'S INTEGRAL FORMULA

We have shown that if \( f(z) \) is analytic at all points interior to and on a closed contour \( C \), then

\[
\oint_C f(z)\,dz = 0.
\]

If \( f(z) \) is analytic everywhere within and on a closed contour \( C \), and \( z_0 \) is any point interior to \( C \), then

\[
f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z_0} \,d\zeta.
\]

This result is called Cauchy's integral formula, and it shows that the value of a function that is analytic in a region is determined throughout the region by its values on the boundary.

By successively differentiating the Cauchy's integral formula with respect to \( z_0 \), we find that

\[
f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \,d\zeta
\]

for \( n=1,2,3,\ldots \).

For reasons of application we usually write

\[
f^{(n)}(z_0) = \frac{(n-1)!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^n} \,dz = \frac{2\pi i f^{(n-1)}(z_0)}{(n-1)!}.
\]

Example:

\[
f^{(n)}(z_0) = \oint_{\Gamma} \frac{e^z + \sin z}{z} \,dz
\]

where \( \Gamma \) is the circle \(|x-2|=3\) traversed once in the counterclockwise direction.

Observe that

\[f(z) = e^z + \sin z\]

is analytic inside and on \( \Gamma \), and that the point \( z_0=0 \) is inside this circle. Hence, by Cauchy's integral formula

\[
\oint_{\Gamma} \frac{e^z + \sin z}{z} \,dz = 2\pi i f(0) = 2\pi i \left|e^0 + \sin 0\right| = 2\pi i.
\]
Example:

\[ \int_C \frac{2z+1}{z(z-1)^2} \, dz \]

where the contour \( C \) is as sketched below

\[ \Gamma_2 \quad \Gamma_1 \]

Notice that integration along \( C \) is equivalent to integrating once around the positively oriented right lobe \( \Gamma_1 \), and then integrating once around the negatively oriented left lobe \( \Gamma_2 \). Expanding by partial fractions, we have

\[ \int_C \frac{2z+1}{z(z-1)^2} \, dz = \int_{\Gamma_1} \frac{(2z+1)/z}{(z-1)^2} \, dz + \int_{\Gamma} \frac{(2z+1)/(z-1)^2}{z} \, dz \]

and applying the derivative form of Cauchy's integral formula,

\[ \int_C \frac{2z+1}{z(z-1)^2} \, dz = \frac{2}{1!} \frac{d}{dz} \left( \frac{2z+1}{z} \right) \bigg|_{z=1} - 2\pi i \int_C \frac{2z+1}{(z-1)^2} \, dz \bigg|_{z=0} = -2\pi i - 2\pi i = -4\pi i. \]