COMPLEX VALUED FUNCTIONS

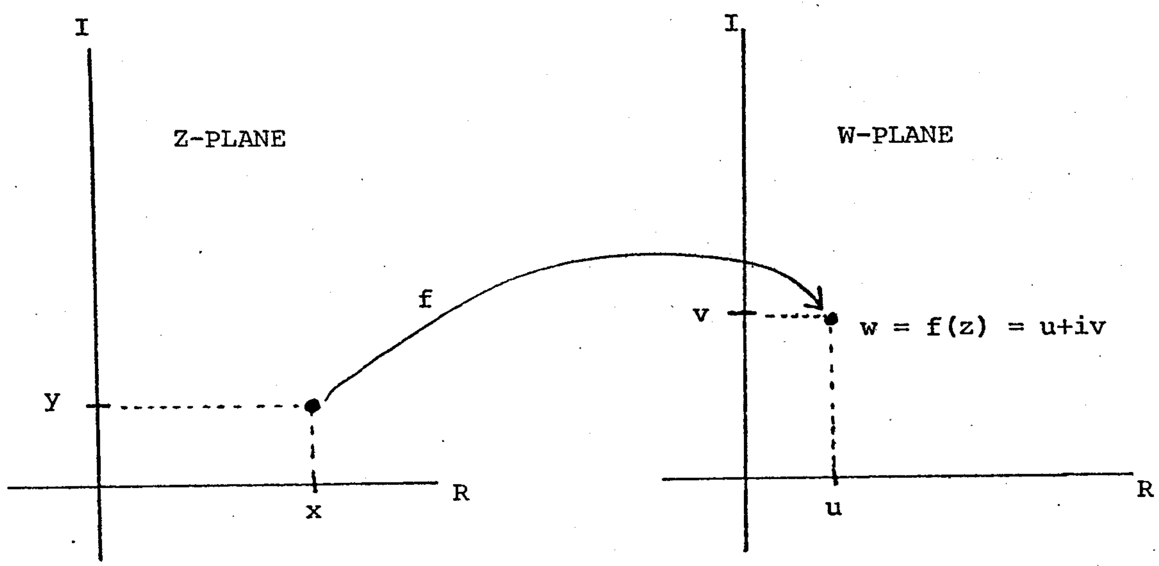
You will recall that we define a complex valued function as follows:

$z = x + iy$ is the complex variable,
f is the complex function,

and

$$w = f(z) = f(x+iy) = u(x,y) + iv(x,y)$$

is viewed as a mapping from the complex z-plane to another plane called the w-plane.



Furthermore, $f(z)$ is called *analytic* at a point z_o if and only if $f(z)$ is differentiable at every point in a neighborhood of z_o. As a more convenient criterion, if u and v are real, single-valued functions of x and y, and, with their first four partial derivatives, are continuous throughout a region of interest, then the *Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

are both necessary and sufficient conditions that $f(z) = u(x,y) + iv(x,y)$ be analytic in that region.

DEFINITE COMPLEX INTEGRALS

Again expressing the complex function f(z) in terms of its real and imaginary parts, we have

$$f(t) = u(t) + iv(t) \quad .$$

Here we have parametrized u and v (and therefore x and y) with respect to the parameter t (usually time in applications). If u and v are continuous, except possibly at a finite number of points where the function, although discontinuous, has both left-hand and right-hand limits, then we say that f(t) is *piecewise continuous*, and we can be assured that the integral

$$\int_a^b f(t)dt$$

exists, that is, it converges. We define the meaning of the above expression as follows:

$$\int_a^b f(t)dt = \int_a^b \{u(t) + iv(t)\}dt = \int_a^b u(t)dt + \int_a^b iv(t)dt$$
$$= \int_a^b u(t)dt + i\int_a^b v(t)dt \quad .$$

Furthermore, it follows from this definition that if k is a complex number, then

$$\int_a^b kf(t)dt = k\int_a^b f(t)dt \quad .$$

That this forms a natural extention of the real case is demonstrated by

$$\text{Re}\int_a^b f(t)dt = \int_a^b \text{Re}\{f(t)\}dt \quad .$$

CONTOURS

A point set S in the complex plane is called a *smooth arc* or a *regular arc* if it is the image of some continuous complex-valued function f(t), $a \le t \le b$, such that

- f(t) has a continuous derivative in the interval,
- $f'(t) \neq 0$ anywhere in the interval, and
- f(t) is one-to-one throughout the interval.

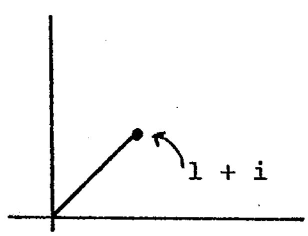
S is called a *smooth closed curve* if it is a smooth arc, but is one-to-one on the interval $a \le t < b$, and $f(b) = f(a)$, and $f'(b) = f'(a)$.

"S is a smooth curve" means precisely that S is either a smooth arc or a smooth closed arc.

Translation: a smooth curve possesses a unique tangent at every point and the tangent varies continuously along the curve. (Therefore a smooth curve has no corners.)

Example:

The function used to describe a curve is somewhat arbitrary. Observe the smooth curve below, consisting of a straight-line segment from (0,0) to (1,1).



Each of the following functions satisfies our definition:

$f_1(t) = t + it \quad (0 \le t \le 1)$
$f_2(t) = (1 - 2t) + (1 - 2t)i \quad (0 \le t \le \frac{1}{2})$
$f_3(t) = \tan(t) + i\tan(t) \quad (0 \le t \le \pi/4)$
$f_4(t) = \sin(t) + i\sin(t) \quad (0 \le t \le \pi/2)$

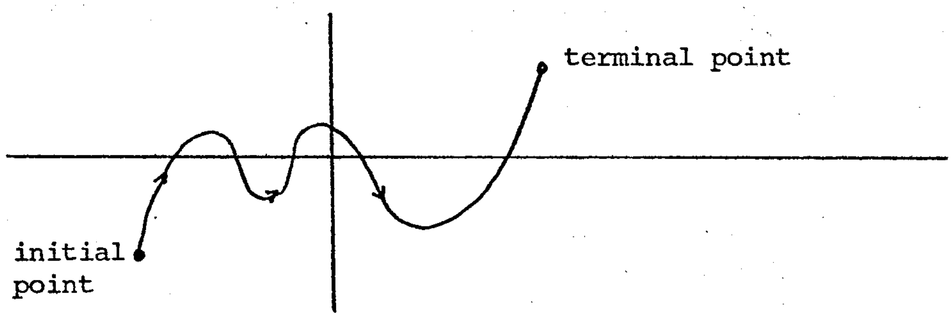
In fact, any function satisfying the conditions of the definition of a smooth arc is called an *admissible parametrization* of S.

One parametrization of any function $y = f(x)$ is trivial; simply let $x = t$ and $y = f(t)$ in the expression $z(t) = x + iy$.

For example, $y = x^3$ for $0 \leq x \leq 1$ can be written as

$$z(t) = t + it^3 \qquad (0 \leq t \leq 1) \quad .$$

A smooth arc, together with an ordering of its points, is called a *directed smooth arc*. In diagrams this is usually denoted by arrows.



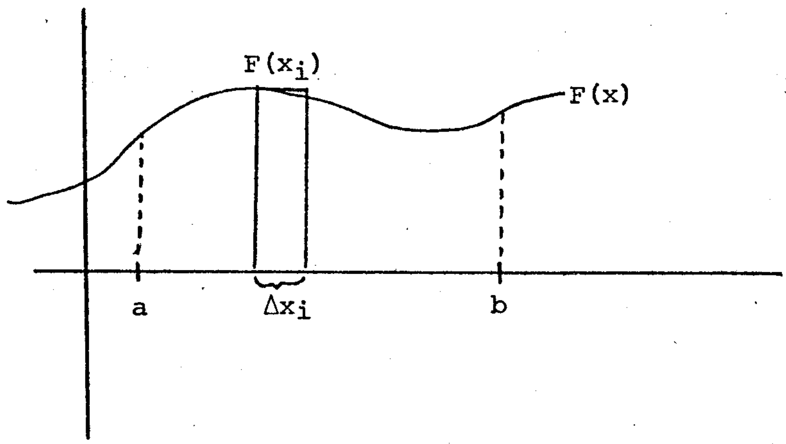
A *Contour* Γ is either a single point z_o or a finite sequence of directed smooth curves $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ such that the terminal point of γ_k coincides with the initial point of γ_{k+1} for each $k = 1, 2, \ldots, n-1$.

If in addition the contour does not cross itself, and its initial and terminal points coincide, then it is called a *simple closed contour*, or, equivalently, a *Jordan contour*, or a *Jordan Curve*.

The Jordan Curve theorem states that any Jordan curve separates the plane into two domains, each having the contour as its boundary. The domain that is bounded is called the *interior*.

CONTOUR INTEGRALS

Recall from the Calculus (of real variables) that the definite integral of a real-valued function F over an interval $a \leq x \leq b$ is established as the limit of a sum of partitions of the area defined by the curve and the x-axis (see figure below).



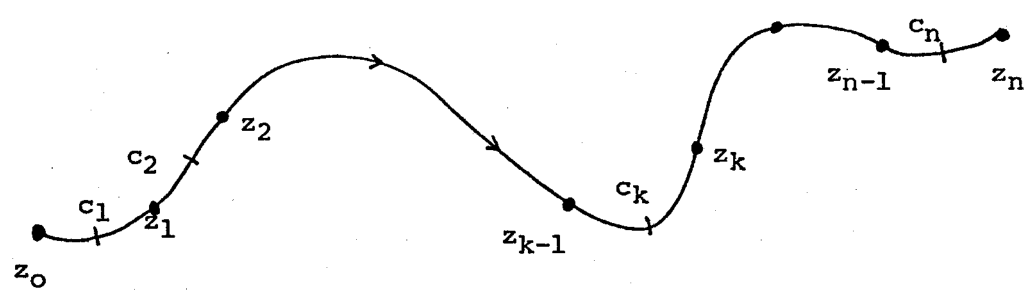
Thus,

$$\int_a^b F(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x_k$$

We can define contour integration in an entirely analogous fashion.

Let the complex-valued function $f(z)$ be defined along a directed smooth curve γ with initial point α and terminal point β. (α can equal β.) For definiteness, assume the points of γ are ordered from α to β .

For some integer n, such that $0 < n < \infty$, define P_n to be a *partition* of γ, that is, a set of points z_o, z_1, ... , z_n, such that $z_o = \alpha$, $z_n = \beta$, and z_{k-1} precedes z_k on the contour, for $k = 1,2, \ldots ,n$. (See the figure below)



If we let $l(z_{k-1}z_k)$ be defined as the length of the arc of γ between points z_{k-1} and z_k, then

$$\mu(P_n) = \max_{1\leq k\leq n} l(z_{k-1}z_k)$$

is called the *mesh* of the partition and is a measure of the "fineness" of the subdivision.

If $c_1, c_2, \ldots, c_n$ are points on γ such that c_1 lies on the arc between z_o and z_1, c_2 lies on the arc between z_1 and z_2, etc., then

$$S(P_n) = f(c_1)(z_1-z_o) + f(c_2)(z_2-z_1) + \cdots + f(c_n)(z_n-z_{n-1})$$

is called a *Riemann sum* for the function f with respect to the partition P_n.

Writing

$$z_k - z_{k-1} = \Delta z_k$$

we reduce the notation to

$$S(P_n) = \sum_{k=1}^{n} f(c_k)(z_k-z_{k-1}) = \sum_{k=1}^{n} f(c_k)\Delta z_k$$

We are now prepared to state the following

Definition. Let f(z) be a complex-valued function defined on the directed smooth curve C. We say that f(z) *is integrable along C* if there exists a complex number L which is the limit of every sequence of Riemann sums $S(P_1), S(P_2), \ldots, S(P_n), \ldots$, corresponding to any sequence of partitions of C satisfying

$$\lim_{n\to\infty} \mu(P_n) = 0, \text{ i.e.,}$$

$$\lim_{n\to\infty} S(P_n) = L \quad \text{whenever} \quad \lim_{n\to\infty} \mu(P_n) = 0 .$$

The constant L is called the *contour integral of f(z) along C*, and we write

$$L = \int_C f(z)dz \quad \text{or} \quad L = \int_C f .$$

Conditions required for integrability are established in the following

Theorem. Let f(z) be a function continuous on the directed smooth curve C. Then if $z = z(t)$, $a \leq t \leq b$, is any admissible parametrization of C consistant with its direction, we have

$$\int_C f(z)dz = \int_a^b f\{z(t)\}z'(t)dt \quad .$$

As expected, integration along a contour is equivalent to summing the results of its constituent curves; formally we have the following

Definition. Suppose C is a contour consisting of the directed smooth curves $(c_1, c_2, \ldots, c_n)$, and let f(z) be a function continuous on C. Then the *contour integral of f(z) along C* is denoted by the symbol

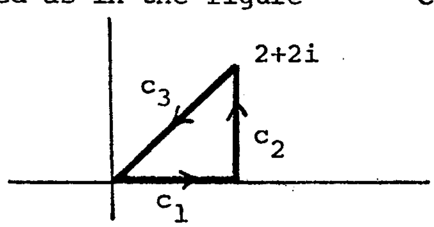
$$\int_C f(z)dz$$

and is defined by

$$\int_C f(z)dz = \int_{c_1} f(z)dz + \int_{c_2} f(z)dz + \cdots + \int_{c_n} f(z)dz \quad .$$

Example:

Compute the contour integral $\int_C \bar{z}^2 dz$, where C is defined as in the figure



We choose the parametrizations:

c_1: $z_1(t) = t$ $\quad(0 \leq t \leq 2)$
c_2: $z_2(t) = 2+ti$ $\quad(0 \leq t \leq 2)$
c_3: $z_3(t) = -t(1+i)$ $\quad(-2 \leq t \leq 0)$

Using

$$\int_C f(z)\,dz = \int_a^b f\left[z(t)\right] z'(t)\,dt$$

we have

$$\int_{C_1} \bar{z}^2 dz = \int_0^2 \overline{z_1(t)}^2 z_1'(t)\,dt = \int_0^2 t^2 dt = \frac{t^3}{3}\Bigg|_0^2 = \frac{8}{3}$$

and

$$\int_{C_2} \bar{z}^2 dz = \int_0^2 \overline{z_2(t)}^2 z_2'(t)\,dt = \int_0^2 (2-ti)^2 i\,dt$$

$$= \frac{i(2-ti)^3}{-3i}\Bigg|_0^2 = -\frac{(2-2i)^3}{3} + \frac{8}{3}$$

and

$$\int_{C_3} \bar{z}^2 dz = \int_{-2}^0 \overline{z_3(t)}^2 z_3'(t)\,dt = \int_{-2}^0 (-t(1-i))^2(-(1+i))\,dt$$

$$= -(1+i)(1-i)^2 \int_{-2}^0 t^2 dt = -(1+i)(1-i)^2\,\frac{8}{3}$$

summing, we have

$$\int_C \bar{z}^2 dz = \frac{8}{3} + \left[\frac{-(2-2i)^3}{3} + \frac{8}{3}\right] + \left[-(1+i)(1-i)^2\,\frac{8}{3}\right]$$

which, after some manipulation, reduces to

$$\int_C \bar{z}^2 dz = \frac{16}{3} + \frac{32}{3}i \quad .$$

When we extend the fundamental theorem of integral calculus to contour integrals and the complex case, the result is a statement regarding "independence of path" of the contour of integration, that is, under certain circumstances, only the endpoints of the contour are relevant to the integration.

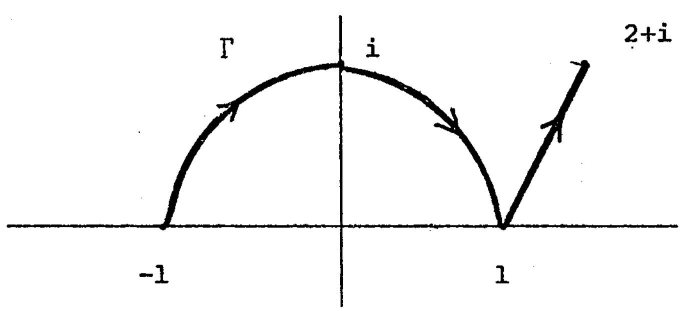
Theorem: If $f(z)$ is continuous throughout a domain Ω and has an antiderivative $F(z)$ throughout Ω, that is, $dF(z)/dz=f(z)$ at all z of Ω, then for any contour Γ lying in Ω, with initial point z_1 and terminal point z_T, we have

$$\int_\Gamma f(z)dz = F(z_T) - F(z_1).$$

NOTE: This amounts to requiring that $F(z)$ is analytic (and hence, continuous) in Ω.

EXAMPLE:

Compute $\int_\Gamma \cos z \, dz$, where Γ is shown in the following figure:



recognizing that $f(z)= \cos z$ has the antiderivative $F(z)=\sin z$ for all z, we apply the theorem immediately:

$$\int_\Gamma \cos z \, dz = \sin z \Big|_{-1}^{2+i} = \sin (2+i) - \sin (-1).$$

As an immediate consequence, we realize that if Γ is a simple closed (Jordan) contour, then its initial and terminal points are the same point; this means that the function will have the same value when evaluated at the upper and lower limits, therefore

$$\int_\Gamma f(z)dz = 0.$$

More formally, we add the following

Corollary. If f(z) is continuous in a domain D and has a antiderivative throughout D, then

$$\int_\Gamma f(z)\,dz = 0$$

for all "loops" Γ lying in D.