COMPLEX VALUED FUNCTIONS

You will recall that we define a complex valued function as follows:

\[ z = x + iy \text{ is the complex variable,} \]
\[ f \text{ is the complex function,} \]
\[ w = f(z) = f(x+iy) = u(x,y) + iv(x,y) \]

is viewed as a mapping from the complex z-plane to another plane called the w-plane.

Furthermore, \( f(z) \) is called analytic at a point \( z_0 \) if and only if \( f(z) \) is differentiable at every point in a neighborhood of \( z_0 \). As a more convenient criterion, if \( u \) and \( v \) are real, single-valued functions of \( x \) and \( y \), and, with their first four partial derivatives, are continuous throughout a region of interest, then the Cauchy-Riemann equations

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\]

are both necessary and sufficient conditions that \( f(z) = u(x,y) + iv(x,y) \) be analytic in that region.
DEFINITE COMPLEX INTEGRALS

Again expressing the complex function \( f(z) \) in terms of its real and imaginary parts, we have

\[
f(t) = u(t) + iv(t) .
\]

Here we have parametrized \( u \) and \( v \) (and therefore \( x \) and \( y \)) with respect to the parameter \( t \) (usually time in applications). If \( u \) and \( v \) are continuous, except possibly at a finite number of points where the function, although discontinuous, has both left-hand and right-hand limits, then we say that \( f(t) \) is piecewise continuous, and we can be assured that the integral

\[
\int_a^b f(t)\,dt
\]

exists, that is, it converges. We define the meaning of the above expression as follows:

\[
\int_a^b f(t)\,dt = \int_a^b \{u(t) + iv(t)\}\,dt = \int_a^b u(t)\,dt + \int_a^b iv(t)\,dt
\]

\[
= \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt .
\]

Furthermore, it follows from this definition that if \( k \) is a complex number, then

\[
\int_a^b kf(t)\,dt = k\int_a^b f(t)\,dt .
\]

That this forms a natural extension of the real case is demonstrated by

\[
\Re\int_a^b f(t)\,dt = \int_a^b \Re\{f(t)\}\,dt .
\]
CONTOURS.

A point set $S$ in the complex plane is called a smooth arc or a regular arc if it is the image of some continuous complex-valued function $f(t)$, $a \leq t \leq b$, such that

- $f(t)$ has a continuous derivative in the interval,
- $f'(t) \neq 0$ anywhere in the interval, and
- $f(t)$ is one-to-one throughout the interval.

$S$ is called a smooth closed curve if it is a smooth arc, but is one-to-one on the interval $a \leq t < b$, and $f(b) = f(a)$, and $f'(b) = f'(a)$.

"$S$ is a smooth curve" means precisely that $S$ is either a smooth arc or a smooth closed arc.

Translation: a smooth curve possesses a unique tangent at every point and the tangent varies continuously along the curve. (Therefore a smooth curve has no corners.)

Example:

The function used to describe a curve is somewhat arbitrary. Observe the smooth curve below, consisting of a straight-line segment from $(0,0)$ to $(1,1)$.

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   1 + i
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Each of the following functions satisfies our definition:

- $f_1(t) = t + it \quad (0 \leq t \leq 1)$
- $f_2(t) = (1 - 2t) + (1 - 2t)i \quad (0 \leq t \leq 1/2)$
- $f_3(t) = \tan(t) + itan(t) \quad (0 \leq t \leq \pi/4)$
- $f_4(t) = \sin(t) + isin(t) \quad (0 \leq t \leq \pi/2)$

In fact, any function satisfying the conditions of the definition of a smooth arc is called an admissible parametrization of $S$.  

One parametrization of any function \( y = f(x) \) is trivial; simply let \( x = t \) and \( y = f(t) \) in the expression \( z(t) = x + iy \).

For example, \( y = x^3 \) for \( 0 \leq x \leq 1 \) can be written as

\[
z(t) = t + it^3 \quad (0 \leq t \leq 1).
\]

A smooth arc, together with an ordering of its points, is called a directed smooth arc. In diagrams this is usually denoted by arrows.

A Contour \( \Gamma \) is either a single point \( z_0 \) or a finite sequence of directed smooth curves \( (\gamma_1, \gamma_2, \ldots, \gamma_n) \) such that the terminal point of \( \gamma_k \) coincides with the initial point of \( \gamma_{k+1} \) for each \( k = 1, 2, \ldots, n-1 \).

If in addition the contour does not cross itself, and its initial and terminal points coincide, then it is called a simple closed contour, or, equivalently, a Jordan contour, or a Jordan Curve.

The Jordan Curve theorem states that any Jordan curve separates the plane into two domains, each having the contour as its boundary. The domain that is bounded is called the interior.
Recall from the Calculus (of real variables) that the definite integral of a real-valued function $F$ over an interval $a \leq x \leq b$ is established as the limit of a sum of partitions of the area defined by the curve and the $x$-axis (see figure below).

Thus,

$$\int_a^b F(x) \, dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k) \Delta x_k$$

We can define contour integration in an entirely analogous fashion.

Let the complex-valued function $f(z)$ be defined along a directed smooth curve $\gamma$ with initial point $\alpha$ and terminal point $\beta$. ($\alpha$ can equal $\beta$.) For definiteness, assume the points of $\gamma$ are ordered from $\alpha$ to $\beta$.

For some integer $n$, such that $0 < n < \infty$, define $P_n$ to be a partition of $\gamma$, that is, a set of points $z_0, z_1, \ldots, z_n$, such that $z_0 = \alpha$, $z_n = \beta$, and $z_{k-1}$ precedes $z_k$ on the contour, for $k = 1, 2, \ldots, n$. (See the figure below)
If we let \( l(z_{k-1}z_k) \) be defined as the length of the arc of \( \gamma \) between points \( z_{k-1} \) and \( z_k \), then

\[
\mu(P_n) = \max_{1 \leq k \leq n} l(z_{k-1}z_k)
\]

is called the **mesh** of the partition and is a measure of the "fineness" of the subdivision.

If \( c_1, c_2, \ldots, c_n \) are points on \( \gamma \) such that \( c_1 \) lies on the arc between \( z_o \) and \( z_1 \), \( c_2 \) lies on the arc between \( z_1 \) and \( z_2 \), etc., then

\[
S(P_n) = f(c_1)(z_1-z_o) + f(c_2)(z_2-z_1) + \cdots + f(c_n)(z_n-z_{n-1})
\]

is called a **Riemann sum** for the function \( f \) with respect to the partition \( P_n \).

Writing

\[
z_k - z_{k-1} = \Delta z_k
\]

we reduce the notation to

\[
S(P_n) = \sum_{k=1}^{n} f(c_k)(z_k-z_{k-1}) = \sum_{k=1}^{n} f(c_k)\Delta z_k
\]

We are now prepared to state the following

**Definition.** Let \( f(z) \) be a complex-valued function defined on the directed smooth curve \( C \). We say that \( f(z) \) is **integrable along \( C \)** if there exists a complex number \( L \) which is the limit of every sequence of Riemann sums \( S(P_1), S(P_2), \ldots, S(P_n), \ldots \), corresponding to any sequence of partitions of \( C \) satisfying

\[
\lim_{n \to \infty} \mu(P_n) = 0, \text{ i.e., } \\
\lim_{n \to \infty} S(P_n) = L \quad \text{whenever} \quad \lim_{n \to \infty} \mu(P_n) = 0.
\]

The constant \( L \) is called the **contour integral of \( f(z) \) along \( C \)**, and we write

\[
L = \int_C f(z) \, dz \quad \text{or} \quad L = \oint_C f \circ d.
\]
Conditions required for integrability are established in the following.

Theorem. Let \( f(z) \) be a function continuous on the directed smooth curve \( C \). Then if \( z = z(t), \ a \leq t \leq b \), is any admissible parametrization of \( C \) consistent with its direction, we have

\[
\int_C f(z) \, dz = \int_a^b f(z(t)) z'(t) \, dt.
\]

As expected, integration along a contour is equivalent to summing the results of its constituent curves; formally we have the following:

Definition. Suppose \( C \) is a contour consisting of the directed smooth curves \( (c_1, c_2, \ldots, c_n) \), and let \( f(z) \) be a function continuous on \( C \). Then the contour integral of \( f(z) \) along \( C \) is denoted by the symbol

\[
\int_C f(z) \, dz
\]

and is defined by

\[
\int_C f(z) \, dz = \int_{c_1} f(z) \, dz + \int_{c_2} f(z) \, dz + \cdots + \int_{c_n} f(z) \, dz.
\]

Example:

Compute the contour integral \( \int_C \bar{z}^2 \, dz \), where \( C \) is defined as in the figure:

We choose the parametrizations:

\[
\begin{align*}
c_1: \quad & z_1(t) = t & (0 \leq t \leq 2) \\
c_2: \quad & z_2(t) = 2 + ti & (0 \leq t \leq 2) \\
c_3: \quad & z_3(t) = -t(1+i) & (-2 \leq t \leq 0)
\end{align*}
\]
Using
\[ \int_{c_1} f(z)\,dz = \int_{a}^{b} f[z(t)]\,z'(t)\,dt \]
we have
\[ \int_{c_1} z^2\,dz = \int_{0}^{2} \frac{z_1(t)^2z_1'(t)}{t^2dt} = \int_{0}^{2} \frac{t^3}{3} \left| \begin{array}{c} 2 \\ 0 \end{array} \right. = \frac{8}{3} \]

and
\[ \int_{c_2} \overline{z}^2\,dz = \int_{0}^{2} \frac{z_2(t)^2z_2'(t)}{(2-ti)2idt} = \int_{0}^{2} \frac{t^3}{3} \left| \begin{array}{c} 2 \\ 0 \end{array} \right. = \frac{8}{3} \]

and
\[ \int_{c_3} \overline{z}^2\,dz = \int_{-2}^{0} \frac{z_3(t)^2z_3'(t)}{(-t(1-i))^2(-1+i)\,dt} = \int_{-2}^{0} t^2\,dt = -(1+i)(1-i)^2 \cdot \frac{8}{3} \]

summing, we have
\[ \int_{c} \overline{z}^2\,dz = \frac{8}{3} + \left[ -\frac{(2-2i)^3}{3} + \frac{8}{3} \right] \quad + \left[ -(1+i)(1-i)^2 \frac{8}{3} \right] \]

which, after some manipulation, reduces to
\[ \int_{c} \overline{z}^2\,dz = \frac{16}{3} + \frac{32}{3} i \]
When we extend the fundamental theorem of integral calculus to contour integrals and the complex case, the result is a statement regarding "independence of path" of the contour of integration, that is, under certain circumstances, only the endpoints of the contour are relevant to the integration.

Theorem: If \( f(z) \) is continuous throughout a domain \( \Omega \) and has an antiderivative \( \Phi(z) \) throughout \( \Omega \), that is, \( \frac{d\Phi(z)}{dz} = f(z) \) at all \( z \) of \( \Omega \), then for any contour \( \Gamma \) lying in \( \Omega \), with initial point \( z_1 \) and terminal point \( z_T \), we have

\[
\int_{\Gamma} f(z) \, dz = \Phi(z_T) - \Phi(z_1).
\]

NOTE: This amounts to requiring that \( \Phi(z) \) is analytic (and hence, continuous) in \( \Omega \).

EXAMPLE:

Compute \( \int_{\Gamma} \cos z \, dz \), where \( \Gamma \) is shown in the following figure:

![Diagram of contour integral](image)

recognizing that \( f(z) = \cos z \) has the antiderivative \( \Phi(z) = \sin z \) for all \( z \), we apply the theorem immediately:

\[
\int_{\Gamma} \cos z \, dz = \sin z \bigg|^{2+i}_{-1} = \sin (2+i) - \sin (-1).\]
As an immediate consequence, we realize that if $\Gamma$ is a simple closed (Jordan) contour, then its initial and terminal points are the same point; this means that the function will have the same value when evaluated at the upper and lower limits, therefore

$$\int_{\Gamma} f(z) \, dz = 0.$$ 

More formally, we add the following

**Corollary.** If $f(z)$ is continuous in a domain $D$ and has a antiderivative throughout $D$, then

$$\int_{\Gamma} f(z) \, dz = 0$$

for all "loops" $\Gamma$ lying in $D$. 