Autoregressive Conditional Heteroskedastic Model (ARCH)

We have looked at ways of modeling a time series $y_t$ by using (past) information on the series up until time $t-1$:

$$E[y_t \mid y_{t-1}, y_{t-2}, \cdots]$$

For example, the zero-mean AR(1) process:

$$y_t = \phi y_{t-1} + \epsilon_t.$$  

Where $y_t$ has the conditional mean of $\phi y_{t-1}$ and an unconditional mean of zero, and $\epsilon_t$ is a white noise process with a fixed variance $\text{var}(\epsilon) = \sigma^2$.

The standard approach to heteroskedasticity is to find an exogenous variable $x_t$ which predicts the variance:

$$y_t = \epsilon_t x_{t-1}.$$  

This requires knowing and specifying the cause of the time-varying variance, and does not recognize that both the conditional means and variances may change over time.

Engle (1982) suggests generalizing so that:

\[ y_t \mid Y_{t-1}, X_t \sim N(g_t, h_t) \]

where

\[ Y_{t-1} = \{y_{t-s}, s \geq 1\} \]
\[ X_t = \{x_{t-s}, s \geq 0\} \]

and \( g_t \) and \( h_t \) are both functions of \( Y_{t-1} \) and \( X_t \).

Let \( z_t \) be some subset of the variables in \((Y_{t-1}, X_t)\), so that:

\[ g_t = z_t' \beta \]

and

\[ h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2, \varepsilon_t = y_t - g_t \]

\[ y_t \mid Y_{t-1}, X_t \sim N(g_t, h_t) \]

These together form the ARCH(q) model.
These can be rewritten by defining:

\[ w_t' = (1, \varepsilon_{t-1}^2, \ldots, \varepsilon_{t-q}^2) \]
\[ \alpha' = (\alpha_0, \alpha_1, \ldots, \alpha_q) \]

so that

\[ y_t \mid Y_{t-1}, X_t \sim N(z_t' \beta, w_t' \alpha). \]

EXAMPLE: AR(1) model for \( y_t \) with ARCH(1) errors.

\[ z_t' \beta = \phi y_{t-1} + \varepsilon_t, \text{ note that } q = 1. \]
\[ y_t = \phi y_{t-1} + \varepsilon_t \]
\[ E[\varepsilon_t \mid E_{t-1}] = 0 \]
\[ \text{var}[\varepsilon_t \mid E_{t-1}] = h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 \]

where \( E_t = \{ \varepsilon_{t-s}, s \geq 0 \} \).

assume that \(|\phi| < 1\), so \( y_t \) is stationary. Also, \( h_t > 0 \) requires \( \alpha_0 > 0 \) and \( \alpha_1 \geq 0 \).

Engle (1982) shows that the unconditional variance of \( \varepsilon_t \) will be finite if \( \alpha_1 < 1 \):

\[ \text{var}(\varepsilon_t) = \sigma^2 = \frac{\alpha_0}{1 - \alpha_1} \]
and the conditional variance of $\varepsilon_t$ is:

$$h_t - \sigma^2 = \alpha_1 (\varepsilon_{t-1}^2 - \sigma^2).$$

The errors $\varepsilon_t$ and $\varepsilon_{t-\tau}$ are not correlated, but they are not independent. The squared errors are related by:

$$\text{var}(\varepsilon_t) = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2.$$

Although $y_t$ is conditionally normal, it is not jointly normal, and neither is its marginal distribution. The marginal distribution of $y_t$ will be symmetric if the conditional distribution of $\varepsilon_t$ is symmetric.

Both of conditional mean and the conditional variance depend on the available information set.

Engle (1982) shows that an ARCH(q) model will have a finite, positive variance:

$$\text{var}(\varepsilon_t) = \frac{\alpha_0}{1 - \sum_{i=1}^{q} \alpha_i}$$

if $\alpha_0 > 0$, $\alpha_1, \ldots, \alpha_q \geq 0$, and if all of the roots of the associated characteristic equation lie outside the unit circle so that $\Sigma \alpha < 1$. 

4
EXTENSIONS OF ARCH


Assumes \( \hat{y}_t \) may be ARIMA. Differences \( y_t \) for stationarity to produce an \( y_t \) which is ARMA.

This produces the conditional variance:

\[
h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \delta_i (y_{t-i} - \bar{y})^2 + \delta_0 (y_t - \varepsilon_t - \bar{y})^2
\]

with \( \delta_i \geq 0 \).

This implies an “ARMA-ARCH” or “ARMACH” model.


Generalizes $h_t$ to

$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i h_{t-i}, \beta_i \geq 0.$$ 

This is called a Generalized ARCH (p,q) process, or simply GARCH(p,q). With $g_t = Z_t' \beta$ it is called a GARCH Regression Model. (Note that q is often quite large.)


Extend to allow the conditional variance to affect the mean, hence ARCH-M (ARCH in the Mean) Model.

$$y_t \mid Y_{t-1}, X_t \sim N(Z_{1t}^' \beta + \delta h_t, h_t^2)$$

$$h_t^2 = w_t' \alpha + z_{2t}' \gamma$$
where the $z_{1t}$ and $z_{2t}$ are two possibly different subsets of the variables in $(Y_{t-1}, X_t)$.


Extend to the more general multivariate cases, and also introduce the Integrated GARCH (IGARCH) Model, arising with a unit root in the GARCH(p,q) process.

**APPLICATIONS**


Consider inflation variance.


Engle and Bollerslev (1986) Point to applications in the foreign exchange market to test long bond returns against Shiller’s variance bounds.

It has also been used to derive pricing relations for financial assets.