Autoregressive Conditional Heteroskedastic Model (ARCH)

We have looked at ways of modeling a time series y_t by using (past) information on the series up until time *t*–1:

$$E\left[y_t \mid y_{t-1}, y_{t-2}, \cdots\right]$$

For example, the zero-mean AR(1) process:

$$y_y = \phi y_{t-1} + \varepsilon_t .$$

Where y_t has the conditional mean of ϕy_{t-1} and an unconditional mean of zero, and ε_t is a white noise process with a fixed variance $\mathrm{var}(\varepsilon) = \sigma^2$.

The standard approach to heteroskedasticity is to find an exogenous variable x_t which predicts the variance:

$$y_t = \varepsilon_t x_{t-1}.$$

This requires knowing and specifying the cause of the time-varying variance, and does not recognize that both the conditional means and variances may change over time.

Engle (1982), "Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of U.K. Inflation," *Econometrica*, 987-1008.

Engle (1982) suggests generalizing so that:

$$y_t \mid Y_{t-1}, X_t \sim N(g_t, h_t)$$

where

$$Y_{t-1} = \{y_{t-s}, s \geq 1\}$$
$$X_t = \{x_{t-s}, s \geq 0\}$$

and g_t and h_t are both functions of Y_{t-1} and X_t.

Let z_t be some subset of the variables in (Y_{t-1} , X_t), so that:

$$g_t = z_t' \beta$$

and

$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2, \; \varepsilon_t = y_t - g_t$$
$$y_t \mid Y_{t-1}, X_t \sim N(g_t, h_t)$$

These together form the ARCH(q) model.

These can be rewritten by defining:

$$w_t' = (1, \varepsilon_{t-1}^2, \ldots, \varepsilon_{t-q}^2)$$

$$\alpha' = (\alpha_0, \alpha_1, \ldots, \alpha_q)$$

so that

$$y_t \mid Y_{t-1}, X_t \sim N(z_t'\beta, w_t'\alpha).$$

EXAMPLE: AR(1) model for *y_t* with ARCH(1) errors.

$$z_t'\beta = \phi y_{t-1} + \varepsilon_t, \text{ note that } q = 1.$$

$$y_t = \phi y_{t-1} + \varepsilon_t$$

$$E[\varepsilon_t \mid E_{t-1}] = 0$$

$$\text{var}[\varepsilon_t \mid E_{t-1}] = h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2$$

$$\text{where } E_t = \{\varepsilon_{t-s}, s \geq 0\}.$$

assume that $|\phi| < 1$, so *y_t* is stationary. Also,
$h_t > 0$ *requires* $\alpha_0 > 0$ and $\alpha_1 \geq 0$.

Engle (1982) shows that the unconditional variance of ε_t will be finite if $\alpha_1 < 1$:

$$\text{var}(\varepsilon_t) = \sigma^2 = \frac{\alpha_0}{1 - \alpha_1}$$

and the conditional variance of ε_t is:

$$h_t - \sigma^2 = \alpha_1(\varepsilon_{t-1}^2 - \sigma^2).$$

The errors ε_t and $\varepsilon_{t-\tau}$ are not correlated, but they are not independent. The squared errors are related by:

$$\text{var}(\varepsilon_t) = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2.$$

Although y_t is conditionally normal, it is not jointly normal, and neither is its marginal distribution. The marginal distribution of y_t will be symmetric if the conditional distribution of ε_t is symmetric.

Both of conditional mean and the conditional variance depend on the available information set.

Engle (1982) shows that an ARCH(q) model will have a finite, positive variance:

$$\text{var}(\varepsilon_t) = \frac{\alpha_0}{1 - \sum_{i=1}^{q} \alpha_i}$$

if $\alpha_0 > 0$, $\alpha_1, \ldots, \alpha_q \geq 0$, and if all of the roots of the associated characteristic equation lie outside the unit circle so that $\Sigma\alpha < 1$.

EXTENSIONS OF ARCH

Weiss, A.A. (1984) "ARMA Models with ARCH Errors," *Journal of Time Series*, 129-143.

Assumes $\hat{y}_t$ **may be ARIMA. Differences** y_t **for stationarity to produce an** y_t **which is ARMA.**

This produces the conditional variance:

$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \delta_i (y_{t-i} - \bar{y})^2 + \delta_0 (y_t - \varepsilon_t - \bar{y})^2$$

with $\delta_i \geq 0$.

This implies an "ARMA-ARCH" or "ARMACH" model.

Bollerslev, T. (1986) "Generalized Autoregressive Conditional Heteroskedasticity," ***Journal of Econometrics*****, 307-27.**

______ ,(1988) "On the Correlation Structure for the Generalized Conditional Heteroskedastic Process," ***Journal of Time Series Analysis*****, 121-32.**

Generalizes h_t to

$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i h_{t-i},\ \beta_i \geq 0.$$

This is called a Generalized ARCH (p,q) process, or simply GARCH(p,q). With $g_t = z_t' \beta$ it is called a GARCH Regression Model. (Note that q is often quite large.)

Engle, R.F., Lilien, D.M., and Robbins, R.P. (1987) "Estimating Time Varying Risk Premia in the Term Structure: The ARCH-M Model." ***Econometrica*** **, 391-408.**

Extend to allow the conditional variance to affect the mean, hence ARCH-M (ARCH in the Mean) Model.

$$y_t \mid Y_{t-1}, X_t \sim N(z_{1t}' \beta + \delta h_t, h_t^2)$$

$$h_t^2 = w_t' \alpha + z_{2t}' \gamma$$

where the z_{1t} and z_{2t} are two possibly different subsets of the variables in (Y_{t-1},X_t).

Engle and Bollerslev (1986) "Modeling the Persistence of Conditional Variances" *Econometric Reviews*, 1-50.

Extend to the more general multivariate cases, and also introduce the Integrated GARCH (IGARCH) Model, arising with a unit root in the GARCH(p,q) process.

APPLICATIONS

Engle (1983) Journal of Money, Credit and Banking, 286-301.
Engle, Hendry, and Trumble (1985), Canadian Journal of Economics.
Pagan et al. (1983) Review of Economic Studies.

Consider inflation varance.

Bollerslev (1987) REStat. Considers time varying risk premia in the term structure of interest rates.

Engle and Bollerslev (1986) Point to applications in the foreign exchange market to test long bond returns against Shiller's variance bounds.

It has also been used to derive pricing relations for financial assets.